Decomposing Graphs Along Separations and Beyond

Tim Planken
TU Bergakademie Freiberg
August 26, 2026

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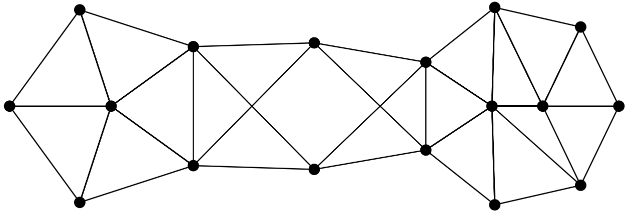

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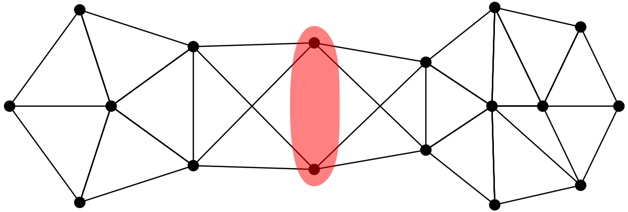

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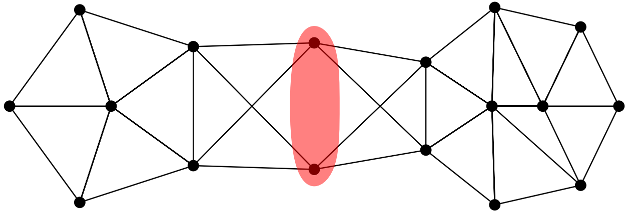

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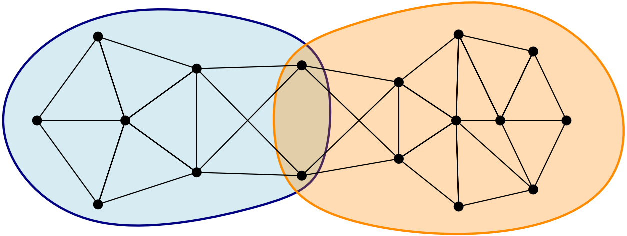

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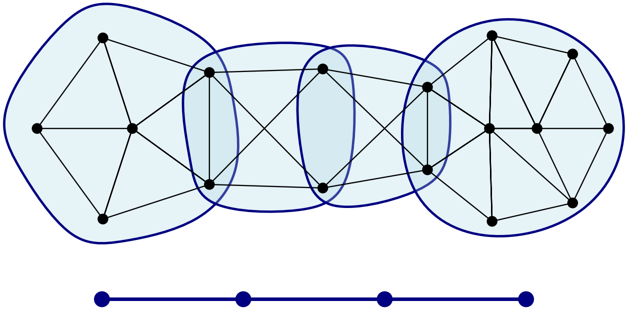

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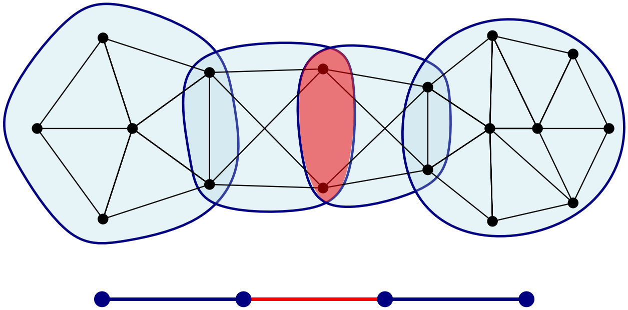

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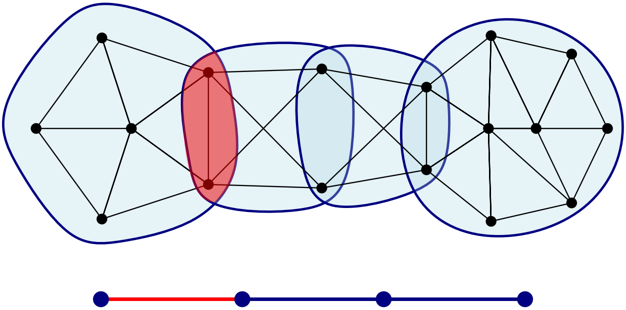

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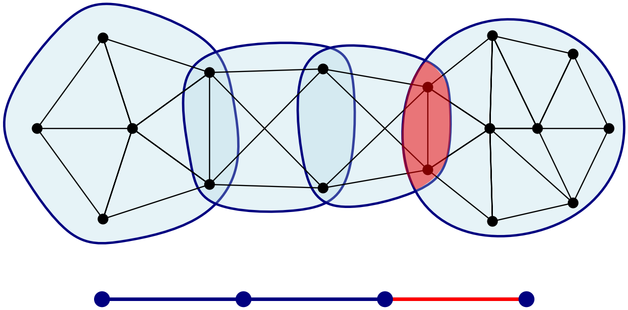

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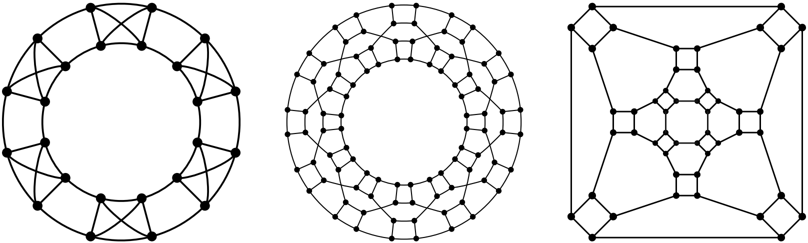
- classifying vertex-transitive finite graphs of low connectivity



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Applications:

- classifying vertex-transitive finite graphs of low connectivity
- reduce problems to high connectivity, e.g.
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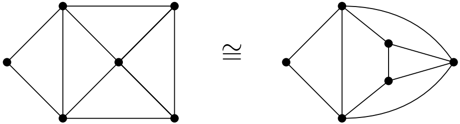
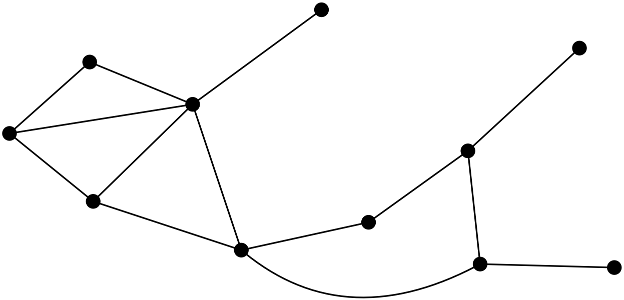
- develop efficient algorithms on graphs, e.g.
  isomorphism testing for planar graphs,
  connectivity augmentation problem



Block-cutvertex decomposition

Given: connected graph G



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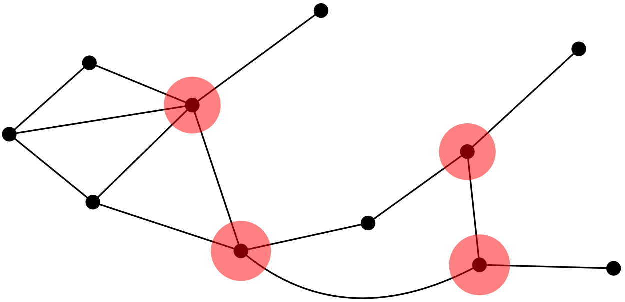

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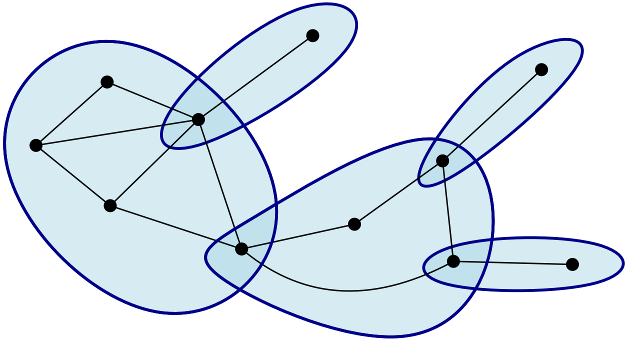

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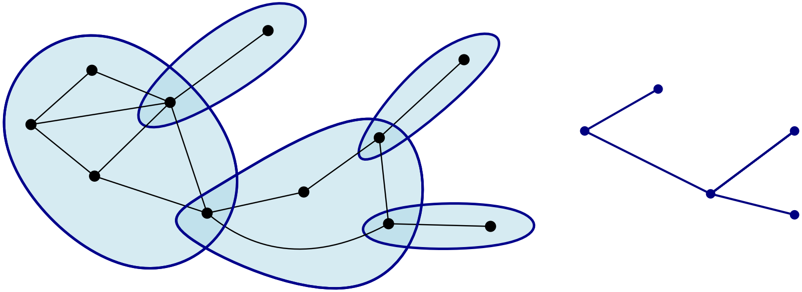
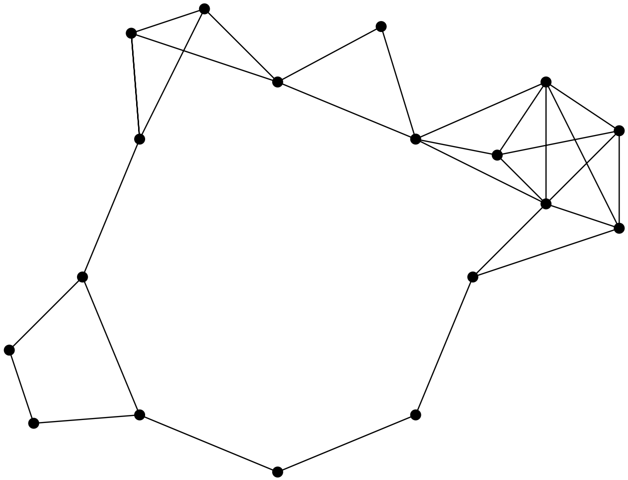


Properties of $\mathcal{T}$:

- building blocks: 2-connected graphs and K_2's
- 'displays' every cutvertex

Tutte's decomposition

Given: 2-connected graph G



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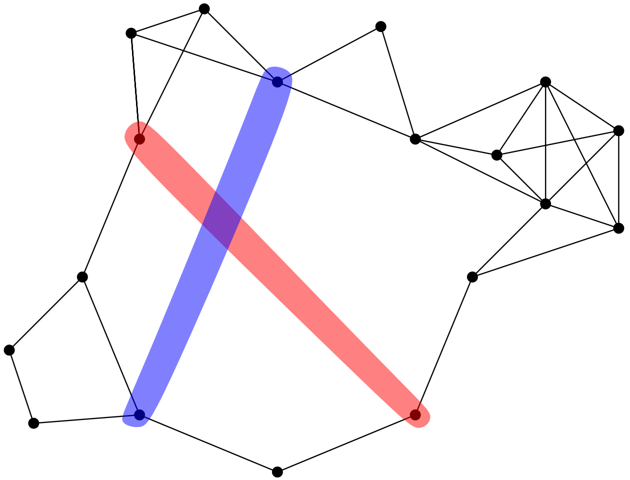

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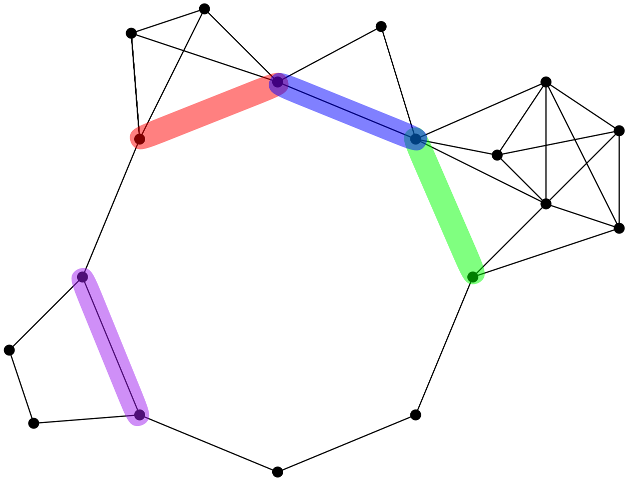

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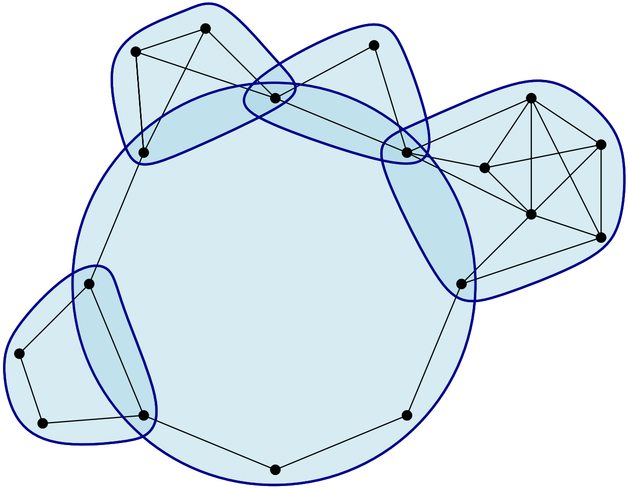

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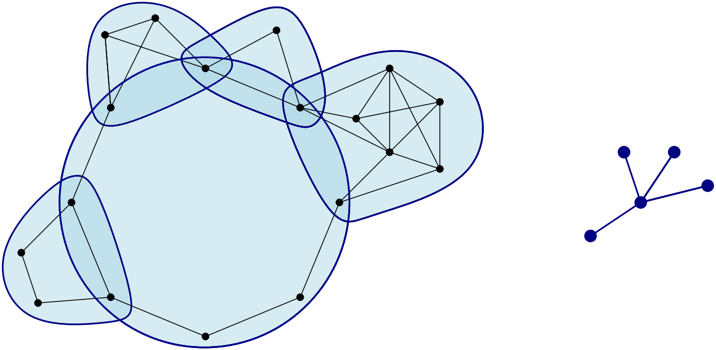
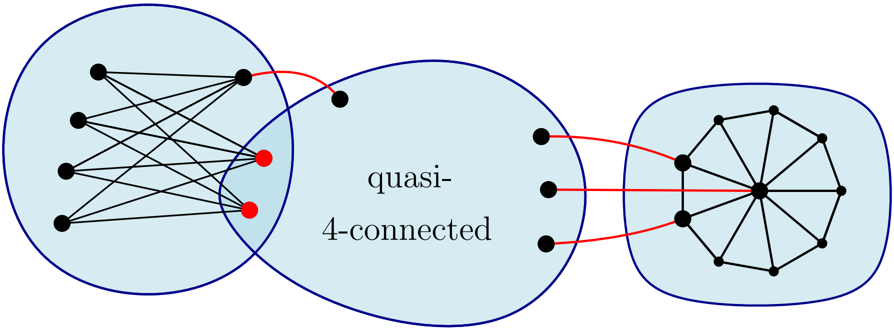


Properties of $\mathcal{T}$:

- building blocks: 3-connected graphs, cycles and K_2's
- 'displays' every 2-separator

Tri-separation decomposition (Carmesin, Kurkofka '23)

Given: 3-connected graph G



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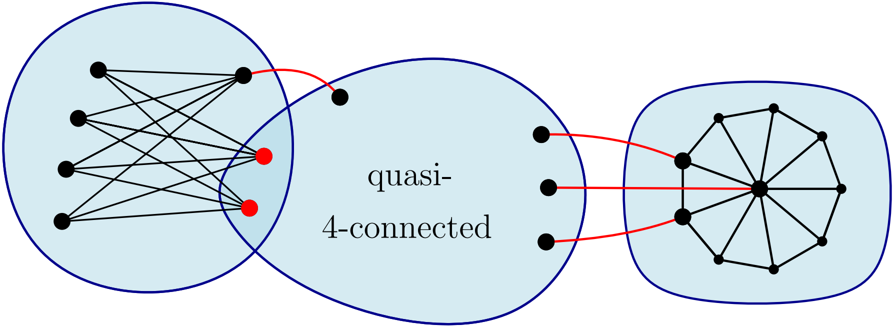


Properties of $\mathcal{T}$:

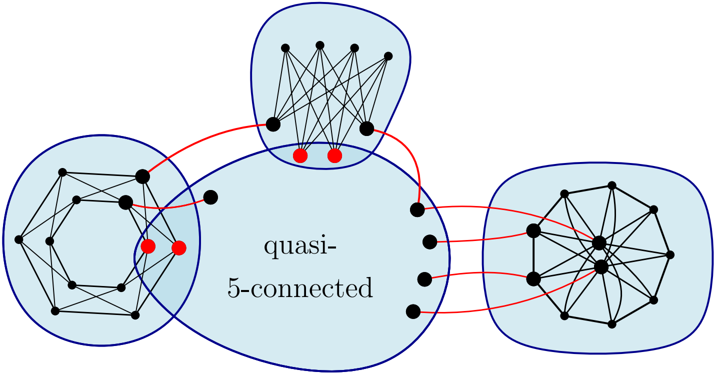
- building blocks: quasi-4-connected graphs, wheels and $K_{3,m}$'s
- 'displays' every 3-separator

Tetra-separation decomposition

Main result (Kurkofka & P. '25)

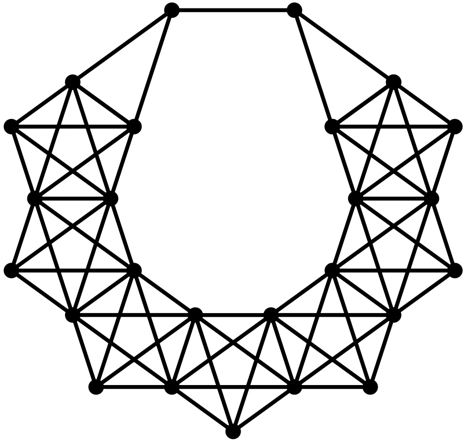
Every 4-connected G decomposes along its totally-nested tetra-separations into parts that are

- quasi-5-connected
- generalised double-wheels
- thickened $K_{4,m}$
- generalised double-cycles



Difficulties in the proof

- "totally-nested"-approach fails even for 3-separators



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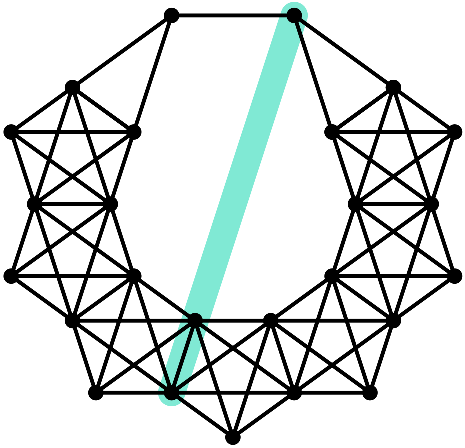

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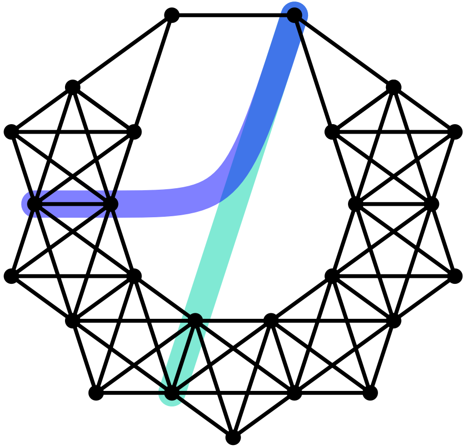

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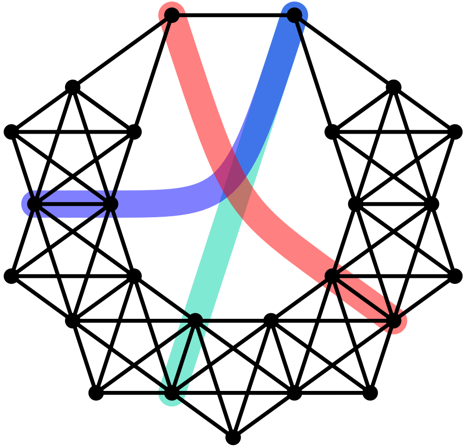

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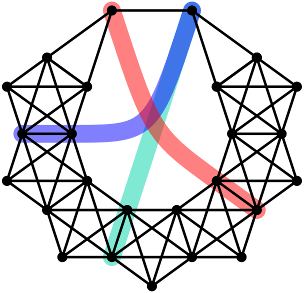 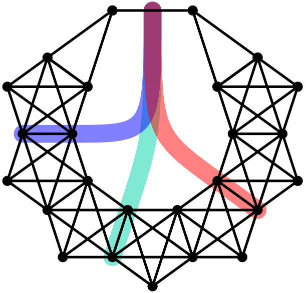

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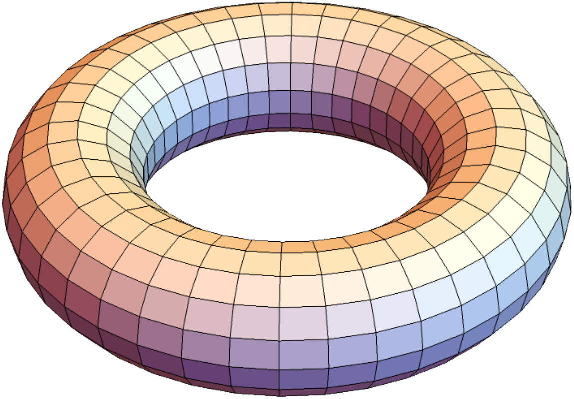

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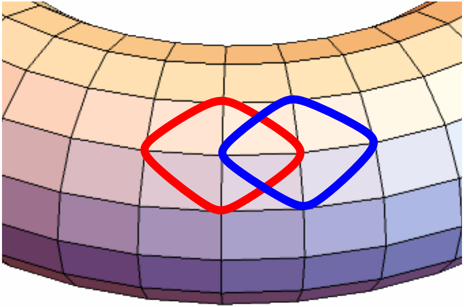

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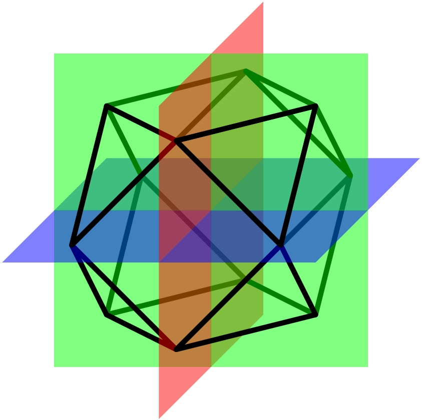

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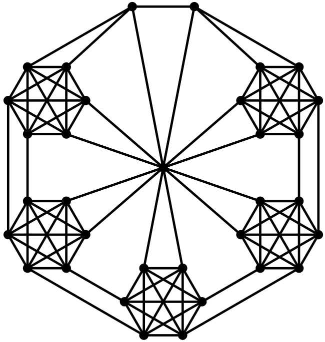

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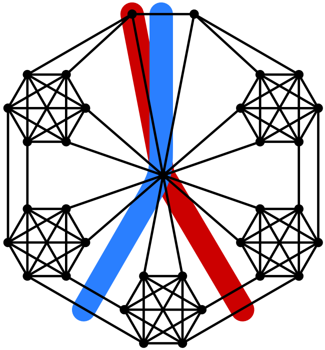
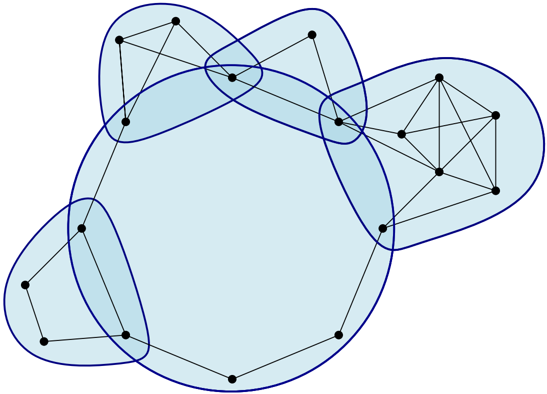
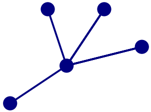


Decomposition for arbitrary vertex-connectivity?

Recap: Tutte's decomposition for 2-connectivity





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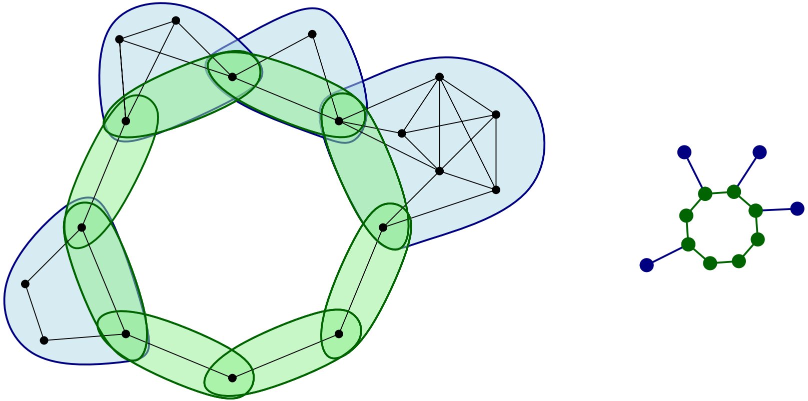

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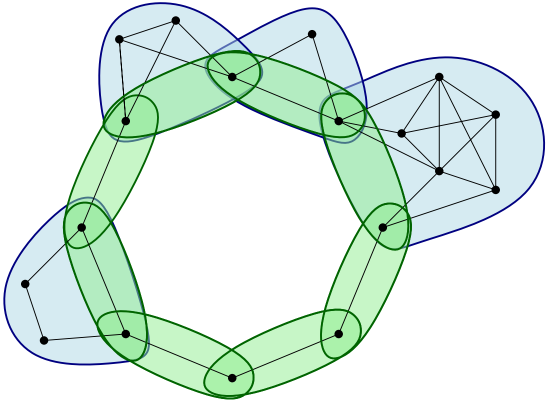

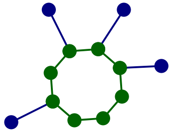

Theorem (Tutte '66)

Every 2-connected G admits a cactus-decomposition $\mathcal{C}$ such that

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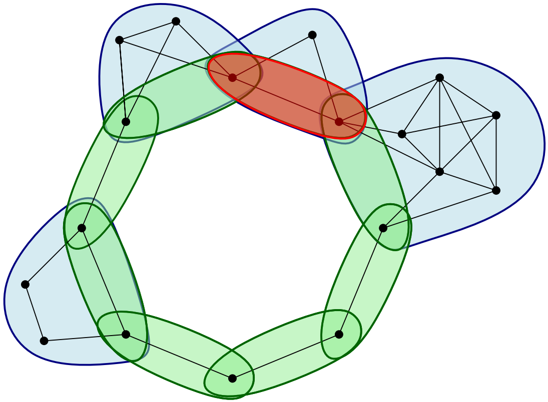

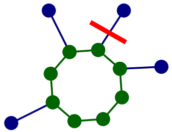

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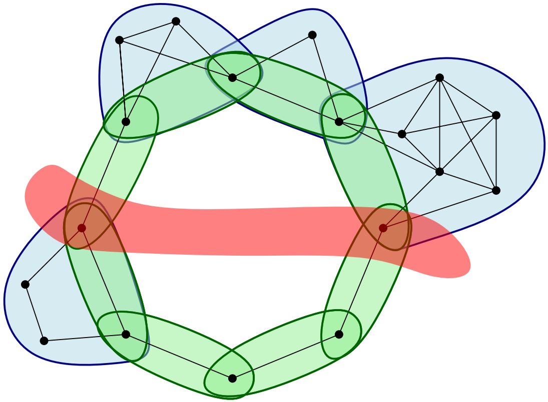

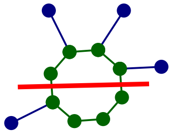

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Theorem (Dinitz, Karzanov, Lomonosov '76)

For every k-edge-connected graph G there is an edge-cactus-decomposition $\mathcal{C}$ such that

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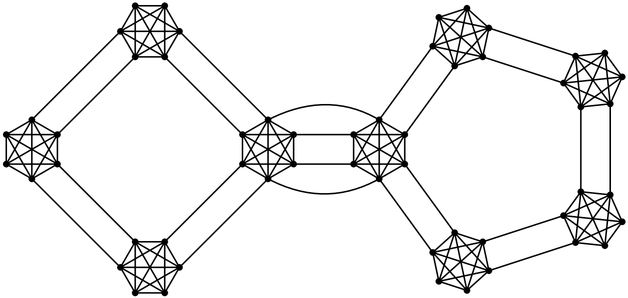

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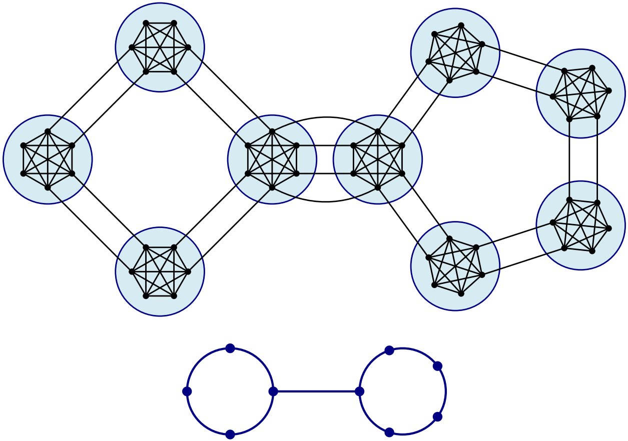

Decomposition for arbitrary vertex-connectivity!

Main result (Bourneuf, Carmesin, Kurkofka & P. 26+)

Every k-connected G admits a cactus-decomposition displaying a set S of k-separations so that *every* k-separation is $c(k)$-close to some separation in S.

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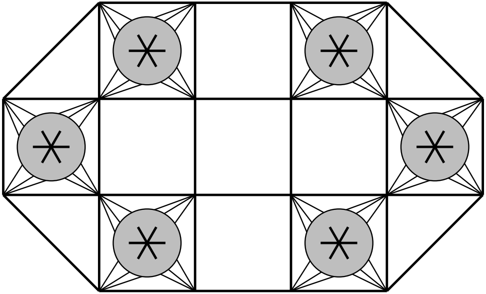

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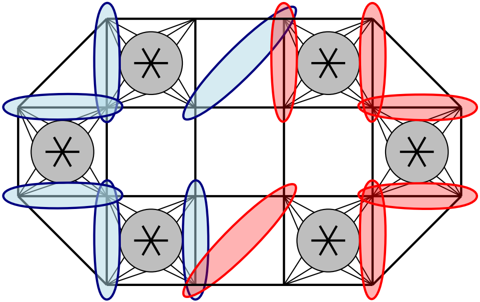

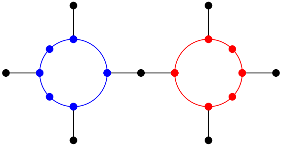

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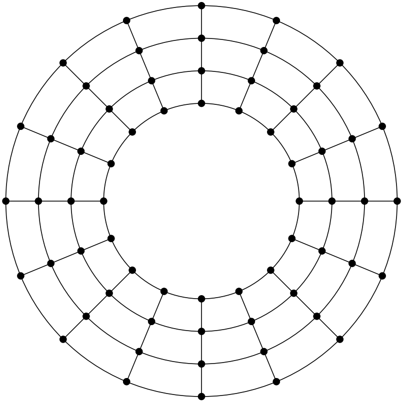

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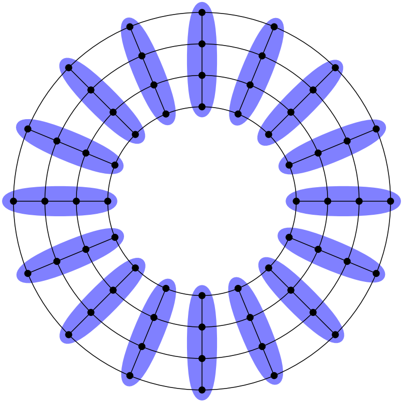

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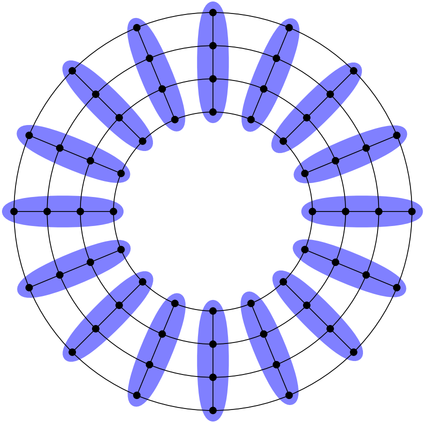

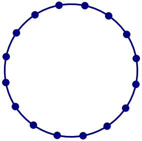

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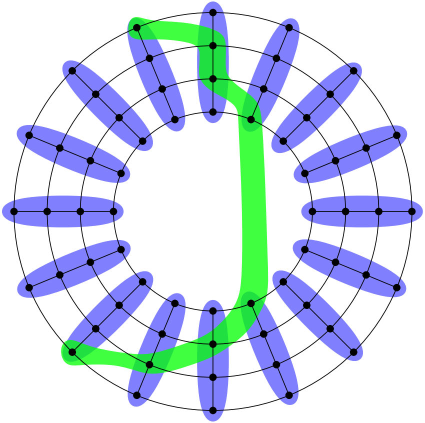

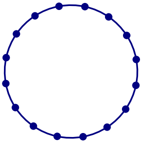

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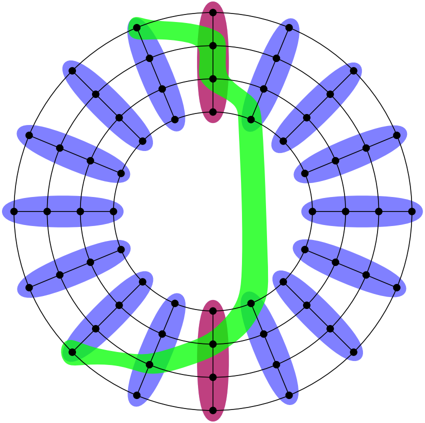

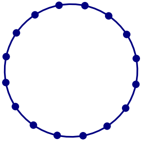

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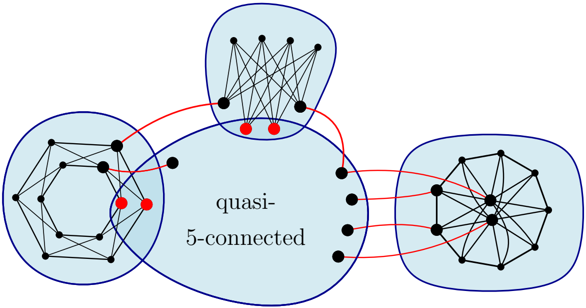

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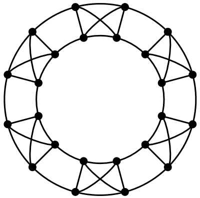
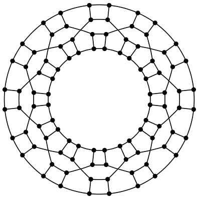
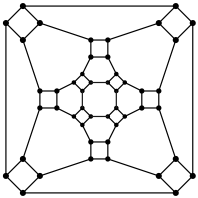

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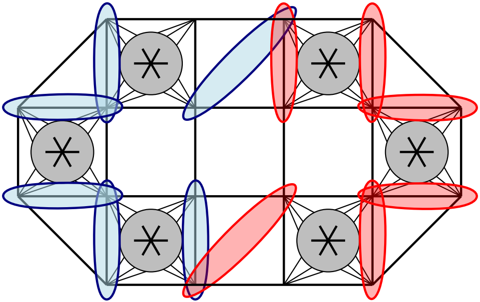

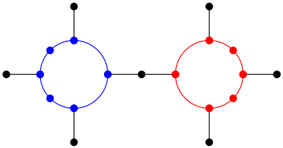

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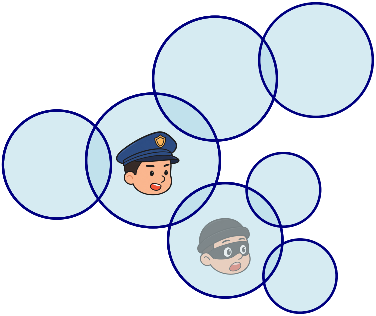

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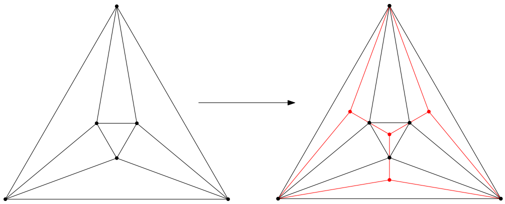

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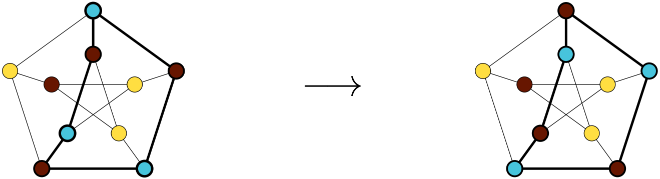

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Thank you!