Karlsruhe Institute of Technology

Shallow Edge Sets in Hypergraphs

Master Thesis of

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Statement of Authorship

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Abstract

Let $H = (V, E)$ be a hypergraph. A subset M of E is *t-shallow* if every vertex $v \in V$ has at most t incident edges that are in M. A subset M of E is *hitting* if every vertex v has an incident edge in M. In this thesis, we consider both shallow edge sets and shallow hitting edge sets. First, we give an equivalence for the existence of shallow hitting edge sets in graphs (i.e. each edge is a 2-subset of V). Moreover, we give a sufficient condition on the minimum vertex degree for the existence of t-shallow hitting edge sets in m-uniform (m-partite) hypergraphs and prove that this condition is almost tight. Then, we provide upper and lower bounds for the smallest $t = t(m)$ such that every m-uniform regular hypergraph has a t-shallow hitting edge set. The proof of the upper bound is constructive and uses the Lovász Local Lemma. Moreover, we consider maximum size shallow edge sets and provide a lower bound for uniform regular hypergraphs and show that this lower bound is tight up to a constant factor. We also provide an explicit construction of m-uniform m-partite regular hypergraphs through projective spaces. This construction improves the lower bound on $t(m)$ and provides an explicit construction of an m-uniform m-partite regular hypergraph with small maximum shallow edge set. Then, we show that deciding whether an m-uniform m-partite regular hypergraph has a t-shallow hitting edge set is $\mathcal{NP}$-complete. Moreover, we use the existence of shallow hitting edge sets in uniform regular hypergraphs to show an upper bound for the polychromatic coloring of the union of strips.

Deutsche Zusammenfassung

Sei $H = (V, E)$ ein Hypergraph. Eine Teilmenge M von E heißt *t-shallow*, wenn jeder Knoten $v \in V$ höchstens t inzidente Kanten aus M hat. Eine Teilmenge M von E heißt *hitting*, wenn jeder Knoten v mindestens eine inzidente Kante aus M hat. In dieser Thesis betrachten wir shallow edge sets und shallow hitting edge sets. Zuerst geben wir eine Äquivalenz für die Existenz von shallow hitting edge sets in Graphen (das heißt, jede Kante ist eine 2-Teilmenge von V) an. Außerdem geben wir hinreichende Bedingungen an den minimalen Knotengrad für die Existenz von t-shallow hitting edge sets in m-uniformen m-partiten Hypergraphen an und beweisen, dass die Bedingungen fast scharf sind. Dann bestimmen wir obere und untere Schranken für das kleinste $t = t(m)$, sodass jeder m-uniforme reguläre Hypergraph ein t-shallow hitting edge set besitzt. Der Beweis der oberen Schranke ist konstruktiv und nutzt das Lovász Local Lemma. Außerdem betrachten wir shallow edge sets maximaler Größe und geben eine untere Schranke für uniforme reguläre Hypergraphen an und zeigen, dass diese untere Schranke scharf ist bis auf einen konstanten Faktor. Zusätzlich geben wir eine explizite Konstruktion von m-uniformen m-partiten regulären Hypergraphen an, die auf projektiven Räumen basiert. Diese Konstruktion verbessert die untere Schranke für $t(m)$ und liefert eine explizite Konstruktion eines m-uniformen m-partiten regulären Hypergraphen mit kleinem shallow edge set maximaler Größe. Dann zeigen wir, dass es $\mathcal{NP}$-schwer ist zu entscheiden, ob ein gegebener m-uniformer m-partiter regulärer Hypergraph ein t-shallow hitting edge set besitzt. Außerdem nutzen wir die Existenz von shallow hitting edge sets in uniformen regulären Hypergraphen, um eine obere Schranke für die polychromatische Färbung von der Vereinigung von Streifen zu zeigen.

Contents

1. Introduction

Matchings and perfect matchings in graphs are an intensively studied subject. A *matching* M in a graph $G = (V, E)$ is a set of edges such that every vertex of G is covered by at most one edge of M. A *perfect matching* M is a matching such that every vertex is covered by exactly one edge of M. One of the best known theorems is Hall's equivalence for the existence of perfect matchings in bipartite graphs. Here, $N(X) = \{v \in V \mid \exists\, u \in X \text{ with } \{u, v\} \in E\}$ denotes the neighborhood of the set X of vertices.

Theorem 1.1 (Hall's Marriage Theorem)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph. Then, G has a matching that covers all vertices in A if and only if $|X| \leq |N(X)|$ for all sets $X \subseteq A$.*

It is a simple consequence of Hall's Theorem that every regular bipartite graph has a perfect matching. A generalization of Hall's Theorem to arbitrary graphs is Tutte's Theorem. Here, the graph $G - S$ is the graph that is made of G by deleting all vertices in S and all edges incident to a vertex in S. Moreover, a component is *odd* if it has an odd number of vertices.

Theorem 1.2 (Tutte's Theorem)**.** *A graph $G = (V, E)$ has a perfect matching if and only if for every set $S \subseteq V$, the size of S is at least the number of odd components in $G - S$.*

Until now, there is no generalization of Hall's Theorem or Tutte's Theorem to hypergraphs. There is an equivalence for the existence of perfect matchings in so-called *balanced* hypergraphs, see [CCKVk96] and [HT02]. On the other hand, there are two ways to extend the sufficient condition of Hall's Theorem to uniform bipartite hypergraphs (here, a hypergraph $H = (V, E)$ is *bipartite* if there exists a partition $V = A \dot{\cup} B$ such that every edge has exactly one vertex in A). Haxell [Hax95] gave a sufficient condition in terms of minimum vertex covers while Aharoni and Haxell [AH00] gave a sufficient condition in terms of maximum matchings. Both theorems are equivalent to Hall's Theorem in the case of bipartite graphs, but do not give a necessary condition in general.

Moreover, if we consider regular m-uniform m-partite hypergraphs (a generalization of regular bipartite graphs), there exists no perfect matching in general. For example, let $H = (V, E)$ be the 3-uniform 3-partite regular hypergraph with vertex set $V = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ and edge set $E = \{x_1y_1z_1, x_1y_2z_2, x_2y_1z_2, x_2y_2z_1\}$, see Figure 1.1. Then, H does not have a perfect matching.

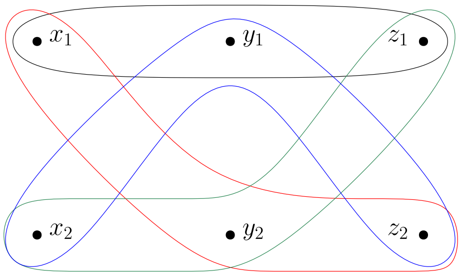

Figure 1.1: A 3-uniform 3-partite regular hypergraph without perfect matching.

Motivated by the study of matchings and perfect matchings in graphs and hypergraphs, we consider t-shallow (hitting) edge sets in this thesis and extend the results for (perfect) matchings to t-shallow (hitting) edge sets. An edge set M is *t-shallow* if every vertex has at most t incident edges that are in M. An edge set M is *hitting* if every vertex is covered by at least one edge of M. Observe that a 1-shallow edge set is a matching and a 1-shallow hitting edge set is a perfect matching. Thus, shallow (hitting) edge sets generalize the concept of (perfect) matchings.

First, we consider shallow hitting edge sets in graphs and bipartite graphs in Section 3.1. Here, we prove some theorems from [BLV78] with the method of alternating paths. It turns out that the method of alternating paths is applicable for t-shallow hitting edge sets in graphs if $t \geq 2$. Moreover, we generalize the results to f-shallow hitting edge sets for a function $f: V \to \mathbb{N} \cup \{0\}$. Here, an edge set M is *f-shallow* if every vertex v is incident to at most $f(v)$ edges in M.

Kühn and Osthus [KO06] started studying sufficient conditions on the minimum vertex degree for the existence of perfect matchings in m-uniform (m-partite) hypergraphs. We extend these results in Section 3.2 to the existence of t-shallow hitting edge sets in m-uniform (m-partite) hypergraphs. Interestingly, the sufficient minimum degree condition in the case $t = 1$ (i.e. we consider perfect matchings) only depends on the number of vertices, whereas the sufficient minimum degree condition in the case $t \geq 2$ decreases with m (i.e. the uniformity of the hypergraph).

As discussed above, regular bipartite hypergraphs always have a perfect matching, while regular m-uniform m-partite hypergraphs do not have perfect matchings in general. In Section 3.3, we construct such a hypergraph that only has t-shallow hitting edge sets with $t \geq \Omega(\log m)$. On the other hand, we provide an upper bound on the smallest $t = t(m)$ such that every regular m-uniform (m-partite) hypergraph has a t-shallow hitting edge set in Section 3.5. Interestingly, $t(m) < \infty$ which implies that every regular m-uniform (m-partite) hypergraph has a t-shallow hitting edge set independently of the number of vertices, number of edges and the regularity of the hypergraph. In fact, we prove that $t(m) \leq \mathcal{O}(m)$. Our proof uses the Lovász Local Lemma, which is stated in Section 3.4. Thus, the proof is constructive, which is explained in Section 3.6 and Section 3.7.

In Chapter 4, we consider maximum t-shallow edge sets (not necessarily hitting), which are generalizations of maximum matchings, in regular m-uniform m-partite hypergraphs. Therefore, we summarize in Section 4.1 the well-known result that every regular m-uniform m-partite hypergraph has a matching of size n/m, where n is the number of vertices per part. In Section 4.2, we extend this result to maximum t-shallow edge sets and prove that $\Omega(nt/m^{1/t})$ is a lower bound for the size of a maximum t-shallow edge set. This result is optimal up to a constant factor, which is proved in Section 4.3 through a construction using combinatorial designs.

There is a construction of a regular m-uniform m-partite hypergraph, namely the truncated projective plane, that only has maximum matchings of size $n/(m-1)$. We summarize these known results in Section 5.1. Motivated by this construction, we consider projective spaces (which are a generalization of projective planes) and introduce the truncated projective space. This simple construction only has t-shallow edge sets of size $nt/(m^{1/t}-1)$, which also shows that the previous result is optimal up to a constant factor.

In Chapter 6, we consider the decision problem of finding t-shallow hitting edge sets in regular m-uniform m-partite hypergraphs. We define that t and m are part of the problem and not part of the input. In Section 3.5 we showed that there is an upper bound $t = t(m) = \mathcal{O}(m)$ such that every such hypergraph has a t-shallow hitting edge set and thus, the decision problem runs in $\mathcal{O}(1)$. In Chapter 6, we show that there exists a bound $t_{\max}(m) = \Omega(\log m)$ such that deciding whether a given regular m-uniform m-partite hypergraph has a t-shallow hitting edge set is $\mathcal{NP}$-complete for $1 \leq t \leq t_{\max}(m)$. Since $t_{\max}(m) \leq t(m)$ for all m, this result is optimal in the sense that $t_{\max}(m)$ matches our lower bound on $t(m)$ up to a constant factor.

In Chapter 7, we consider an application of shallow hitting edge sets to the problem of coloring a geometric hypergraph polychromatically. A *polychromatic k-coloring* of a hypergraph is a k-coloring of the vertices such that each edge receives all k colors. A geometric hypergraph is defined by a finite set V of points in $\mathbb{R}^d$ and a set $\mathcal{R}$ of ranges (i.e. each range is a subset of $\mathbb{R}^d$). A subset e of points builds an edge in the hypergraph if it is captured by a range $R \in \mathcal{R}$, i.e. $V \cap R = e$. In this chapter, we consider axis-aligned strips in d dimensions and improve a known bound from [ACC$^+$11] using the theorem of Section 3.5.

2. Preliminaries

2.1 Notation

By $[n]$, we denote the set of all positive integers up to n, i.e. $[n] = \{1, 2, \ldots, n\}$. For $n = 0$, we define $[0]$ to be the empty set. For a set V and a non-negative integer k, we denote by $\binom{V}{k}$ the set of all k-element subsets of V.

A *multiset* $A = (X, \mu)$ is a set of elements X with assigned multiplicities $\mu\colon X \to \{0, 1, 2, \ldots\}$. If $X = \{x_1, x_2, \ldots, x_n\}$ is the underlying set of elements, we write

$$A = \{\underbrace{x_1, \ldots, x_1}_{\mu(x_1)\text{ times}}, \underbrace{x_2, \ldots, x_2}_{\mu(x_2)\text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{\mu(x_n)\text{ times}}\}$$

or

$$A = \{\mu(x_1) \cdot x_1, \mu(x_2) \cdot x_2, \ldots, \mu(x_n) \cdot x_n\}$$

for the multiset A over X with multiplicities $\mu\colon X \to \{0, 1, 2, \ldots\}$. The *size* $|A|$ of a multiset A is the sum of the multiplicities of the elements in X. For example, the multiset $A = \{x, x, y, y, y\} = \{2 \cdot x, 3 \cdot y\}$ has size $|A| = 5$.

2.2 Graph Definitions

A *hypergraph* H is a pair $H = (V, E)$ with a finite set V and a multiset E of non-empty subsets of V. The elements in V are called vertices and the elements in E are called edges or hyperedges. For a vertex v, we denote by $N(v)$ the set of neighboring vertices of v, i.e. the set of all vertices $u \in V$ with $u \neq v$ such that there exists a hyperedge $e \in E$ with $\{u, v\} \subseteq e$. For a set X of vertices, we denote by $N(X)$ the set of all neighbors of vertices in X, i.e. $N(X) = \cup_{v \in X} N(v)$. For a vertex v, the multiset of incident edges, denoted by $\mathrm{Inc}(v)$, is the set of all edges e with $v \in e$. Throughout this thesis, we assume $\mathrm{Inc}(v) \neq \emptyset$ for all vertices v. The *degree* $\deg(v)$ of a vertex v is the size of $\mathrm{Inc}(v)$. For a subset $F \subseteq E$ of edges, the *degree* $\deg_F(v)$ *of* v *in* F is the size of $F \cap \mathrm{Inc}(v)$. The *maximum degree* of the hypergraph H, denoted by $\Delta(H)$, is the maximum degree of a vertex v in H. Analogous, the *minimum degree* of the hypergraph H, denoted by $\delta(H)$, is the minimum degree of a vertex v in H.

A hypergraph H is said to be *m-uniform*, for a positive integer m, if every hyperedge contains exactly m vertices. A *graph* is a 2-uniform hypergraph. A hypergraph H is said to be *r-regular*, for a non-negative integer r, if every vertex $v \in V$ has degree exactly r. A hypergraph H is said to be *p-partite*, for a positive integer p, if there exists a partition $V = V_1 \dot{\cup} V_2 \dot{\cup} \ldots \dot{\cup} V_p$ into disjoint subsets $V_1, V_2, \ldots, V_p$ of V such that $|e \cap V_i| \leq 1$ for all edges $e \in E$ and all $i = 1, \ldots, p$. In this case, we write $H = (V_1 \dot{\cup} \cdots \dot{\cup} V_p, E)$. The sets $V_1, \ldots, V_p$ are called *parts* or *sides* of H.

A *matching* M is a subset of edges such that $\deg_M(v) \leq 1$ for all vertices $v \in V$. A matching M is called a *perfect matching* if $\deg_M(v) = 1$ for all vertices $v \in V$. A subset $M \subseteq E$ of edges is a *hitting edge set* if $\deg_M(v) \geq 1$ for all vertices $v \in V$. For a positive integer t, an edge set M is *t-shallow* if $\deg_M(v) \leq t$ for all vertices v. For a given edge set M, the least integer t such that M is t-shallow is called *shallowness of M*. Observe that a 1-shallow edge set is a matching and a 1-shallow hitting edge set is a perfect matching. Among others, we consider t-shallow hitting edge sets M, meaning that $1 \leq \deg_M(v) \leq t$ for all vertices v.

An *independent set* is a set V' of vertices such that each edge contains at most one vertex of V', i.e. $|e \cap V'| \leq 1$ for all edges e. A subset $V' \subseteq V$ of vertices is a *hitting set* or *vertex cover* if every edge $e \in E$ contains at least one vertex of V'. An equivalent term for vertex cover is *transversal*. For a positive integer t, a set $V' \subseteq V$ is *t-shallow* if $|e \cap V'| \leq t$ for all edges $e \in E$. For a given set V', the least integer t such that V' is t-shallow is called *shallowness of V'*. By these definitions, a t-shallow vertex cover is a set V' of vertices such that $1 \leq |e \cap V'| \leq t$ for all edges e.

Let $H = (V, E)$ be a hypergraph with vertex set $V = \{v_1, \ldots, v_n\}$ and the multiset of edges $E = \{e_1, \ldots, e_k\}$. The *incidence matrix* A of H is an $(n \times k)$-matrix with $A_{i,j} = 1$ if $v_i \in e_j$, otherwise $A_{i,j} = 0$. The hypergraph H^* is a *dual hypergraph* of H if it has the incidence matrix A^T. Note that, by this definition, $(H^*)^*$ is isomorphic to H. Moreover, observe that all dual hypergraphs of a hypergraph H are isomorphic. Thus, we may also say that H^* is *the* dual hypergraph of H. Observe that H is m-uniform if and only if H^* is m-regular.

3. Shallow Hitting Edge Sets

In this chapter, we first analyse bipartite graphs and generalize Hall's Theorem to shallow hitting edge sets in Section 3.1. Moreover, we show that the theorem can be generalized to graphs that are not bipartite. In Section 3.2, we provide a sufficient condition on the minimum vertex degree of m-uniform m-partite hypergraphs for the existence of t-shallow hitting edge sets. Moreover, we show that the condition is tight by providing a construction of an m-uniform m-partite hypergraph that has large minimum vertex degree but does not have a t-shallow hitting edge set. Then, we provide a lower and an upper bound on the least integer $t = t(m)$ such that every m-uniform m-partite regular hypergraph contains a t-shallow hitting edge set. We show that

$$t(m) \leq \mathrm{e}m \cdot (1 + o(1))$$

is an upper bound and

$$t(m) \geq \lfloor (1 + \log_2 m)/2 \rfloor$$

is a lower bound for $t(m)$. For the lower bound, we provide an explicit construction in Section 3.3. This bound will be improved in Chapter 5 by the factor of 2. For the upper bound, we need the Lovász Local Lemma, which is stated in Section 3.4. In Section 3.5, we prove the upper bound using the Lovász Local Lemma. In Section 3.6, we state the Constructive Lovász Local Lemma and use it in Section 3.7 to prove the existence of a randomized algorithm that outputs a t-shallow hitting edge set for $t(m) = \mathrm{e}m(1 + o(1))$ in expected polynomial time.

Additionally, we generalize the results to hypergraphs that are not far from being regular. For a real number $\mu \geq 1$, we say that a hypergraph $H = (V, E)$ with minimum degree $\delta(H) \geq 1$ is *μ-near regular* if $\Delta(H)/\delta(H) \leq \mu$. Thus, a 1-near regular hypergraph is a regular hypergraph. In the following fact, we provide an equivalent characterization of μ-near regular hypergraphs.

Fact 3.1. *Let $H = (V, E)$ be a hypergraph and $0 \leq \epsilon < 1$. Then, H is μ-near regular for*

$$\mu = \frac{1 + \epsilon}{1 - \epsilon} \tag{3.1}$$

if and only if there exists a real number r such that $(1 - \epsilon)r \leq \deg(v) \leq (1 + \epsilon)r$ for all vertices $v \in V$.

Proof. Let H be a μ-near regular hypergraph with $\Delta = \Delta(H)$ and $\delta = \delta(H)$. Rewriting Equation 3.1 gives $\epsilon = (\mu - 1)/(\mu + 1)$. Let r be the arithmetic mean of Δ and δ, that is $r = (\Delta + \delta)/2$. Then,

$$\deg(v) \geq \delta = \frac{\Delta + \delta}{2} - \frac{\Delta - \delta}{2} = \left(1 - \frac{\Delta - \delta}{\Delta + \delta}\right) r = \left(1 - \frac{\mu - 1}{\mu + 1}\right) r = (1 - \epsilon) r \quad \text{and}$$
$$\deg(v) \leq \Delta = \frac{\Delta + \delta}{2} + \frac{\Delta - \delta}{2} = \left(1 + \frac{\Delta - \delta}{\Delta + \delta}\right) r = \left(1 + \frac{\mu - 1}{\mu + 1}\right) r = (1 + \epsilon) r$$

for all vertices v in V.

On the other hand, assume that $(1 - \epsilon) r \leq \deg(v) \leq (1 + \epsilon) r$ for all vertices v in V where ϵ satisfies Equation 3.1. Then,

$$\frac{\Delta}{\delta} \leq \frac{(1 + \epsilon) r}{(1 - \epsilon) r} = \mu \ .$$

□

3.1 An Equivalence for Graphs

In this section, we show some results from [BLV78] with the method of alternating paths. Let $G = (A \dot\cup B, E)$ be a bipartite graph with $V = A \dot\cup B$, and let $f \colon V \to \mathbb{N}$ be a function with $f(v) \geq 1$ for all vertices v. We say that an edge set M is an *f-shallow edge set* if $\deg_M(v) \leq f(v)$ holds for all $v \in V$. Let $M \subseteq E$ be a subset of edges. A vertex v in G is said to be *covered* by M if $\deg_M(v) > 0$. An *alternating path* is a path $p = (v_0, v_1, \ldots, v_k)$ for some k such that $\{v_i, v_{i+1}\} \notin M$ for all even integers i with $0 \leq i < k$ and $\{v_i, v_{i+1}\} \in M$ for all odd integers i with $0 \leq i < k$. For a function $f \colon V \to \mathbb{N}$, an *$f$-augmenting path* $p = (v_0, v_1, \ldots, v_k)$ is an alternating path such that v_0 is not covered by M and $\deg_M(v_k) < f(v_k)$ if k is odd and $\deg_M(v_k) > 1$ if $k \geq 2$ is even. Note that, if p is an f-augmenting path, we can define an edge set M' by

$$M' = (M \cup \{\{v_i, v_{i+1}\} \mid i \text{ even}, 0 \leq i < k\}) \setminus \{\{v_i, v_{i+1}\} \mid i \text{ odd}, 0 \leq i < k\} \ .$$

Then, $\deg_{M'}(v) = \deg_M(v)$ for all vertices $v \in (A \cup B) \setminus \{v_0, v_k\}$ and $\deg(v_0) = 1$ and $1 \leq \deg(v_k) \leq f(v_k)$. Thus, if M is an f-shallow edge set, then M' is an f-shallow edge set covering more vertices than M.

In the Theorem 3.3, we state an equivalence for the existence of f-shallow hitting edge sets in bipartite graphs. This theorem generalizes Hall's Theorem, which is for the case $f(v) = 1$ for all vertices v. Recall that $N(X)$ is the set of all neighbors of vertices in X.

Theorem 3.2 (Hall's Marriage Theorem [Hal87])**.** *Let $G = (A \dot\cup B, E)$ be a bipartite graph. Then, G has a matching that covers all vertices in A if and only if $|X| \leq |N(X)|$ for all sets $X \subseteq A$.*

Theorem 3.3 ([BLV78])**.** *Let $G = (A \dot\cup B, E)$ be a bipartite graph with $V = A \dot\cup B$, and let $f \colon V \to \mathbb{N}$ be a function with $f(v) \geq 1$ for all vertices v. Then, G has an f-shallow hitting edge set if and only if*

$$|X| \leq \sum_{v \in N(X)} f(v)$$

holds for all sets $X \subseteq A$ and all sets $X \subseteq B$.

Proof. Let M be an f-shallow hitting edge set in G and let X be an arbitrary subset of A. The case $X \subseteq B$ works analogous. Let $M' \subseteq M$ be the set of all edges in M which have a vertex in X. Now, we double count the set M'. On the one hand, $|M'| \geq |X|$ since every vertex in X is covered at least once. On the other hand, $|M'| \leq \sum_{v \in N(X)} f(v)$ since every vertex in $N(X)$ is covered at most $f(v)$ times by an edge in M'. In total, we get $|X| \leq \sum_{v \in N(X)} f(v)$.

Now, assume that G has no f-shallow hitting edge set. Let M be an f-shallow edge set that maximizes the number of covered vertices and let $u \in V$ be an uncovered vertex. Assume without loss of generality $u \in A$. Consider all alternating paths starting in u. If such an alternating path ends in a vertex $v \in B$ with $\deg_M(v) < f(v)$ or in a vertex $v \in A$ with $\deg_M(v) > 1$, then we have an f-augmenting path and thus, we can obtain an f-shallow edge set that covers more vertices than M, a contradiction. In particular, every maximal alternating path ends in a vertex v in A. Thus, if $(u, v_1, v_2, \ldots, v_k)$ is the vertex sequence of a maximal alternating path, then k is even, $\deg_M(v_i) = f(v_i)$ for all odd integers i with $1 \leq i < k$ and $\deg_M(v_i) = 1$ for all even integers i with $1 < i \leq k$. Let X be the set of all vertices v in A such that there exists an alternating path starting in u that contains v. Let $M' \subseteq M$ be the set of all edges in M that have a vertex in X. Then, $|M'| = |X| - 1$ since every vertex in $X \setminus \{u\}$ has degree 1 in M' and u has degree 0 in M'. On the other hand, $|M'| = \sum_{v \in N(X)} f(v)$ since each vertex v in $N(X)$ has degree $f(v)$ in M'. In total, we get

$$|X| > \sum_{v \in N(X)} f(v) \; .$$

The case that there exists an uncovered vertex $v \in B$ works analogous. □

Let t be a positive integer. By setting $f(v) = t$ for all vertices v, we obtain the following corollary.

Corollary 3.4 ([BLV78]). *Let $G = (A \dot\cup B, E)$ be a bipartite graph and let t be a positive integer. Then, G has a t-shallow hitting edge set if and only if*

$$|X| \leq t|N(X)|$$

holds for all sets $X \subseteq A$ and all sets $X \subseteq B$.

Moreover, we obtain the following corollary for μ-near regular bipartite graphs.

Corollary 3.5. *Let $G = (A \dot\cup B, E)$ be a bipartite μ-near regular graph. Then, G has a $\lceil \mu \rceil$-shallow hitting edge set.*

Proof. Let $\delta = \delta(G)$ be the minimum degree and $\Delta = \Delta(G)$ be the maximum degree of G. We prove this corollary by applying Corollary 3.4. Let X be a set of vertices in one part, without loss of generality $X \subseteq A$, and $N(X)$ be the set of neighboring vertices. We count the set of edges that are incident to a vertex in X in two ways. On the one hand, $|\operatorname{Inc}(X)| \geq \delta \cdot |X|$ since every vertex in X has at least δ incident edges. On the other hand, $|\operatorname{Inc}(X)| \leq \Delta \cdot |N(X)|$ since every vertex v in $N(X)$ has at most Δ edges in $\operatorname{Inc}(X)$. Thus, we have

$$|X| \leq \frac{\Delta}{\delta} \cdot |N(X)| \leq \mu \cdot |N(X)| \leq \lceil \mu \rceil \cdot |N(X)| \; .$$

The case $X \subseteq B$ works analogous. By Corollary 3.4, G has a $\lceil \mu \rceil$-shallow hitting edge set. □

In the next theorem, we generalize Theorem 3.3. For a vertex v in a graph G and a function $c\colon E \to \mathbb{N} \cup \{0\}$, we define

$$c(v) = \sum_{e \in \mathrm{Inc}(v)} c(e) \ .$$

Theorem 3.6. *Let $G = (A \dot\cup B, E)$ be a bipartite graph with $V = A \dot\cup B$, and let $f, g\colon V \to \mathbb{N}$ be functions with $1 \leq g(v) \leq f(v)$ for all vertices v. Then, there exists a function $c\colon E \to \mathbb{N} \cup \{0\}$ with*

$$g(v) \leq c(v) \leq f(v) \tag{3.2}$$

for all vertices v if and only if

$$\sum_{v \in X} g(v) \leq \sum_{v \in N(X)} f(v) \tag{3.3}$$

holds for all sets $X \subseteq A$ and all sets $X \subseteq B$.

Proof. Let $c\colon V \to \mathbb{N} \cup \{0\}$ be a function that satisfies Equation 3.2 for all vertices v in V. Let X be an arbitrary subset of A. The case $X \subseteq B$ works analogous. We prove that Equation 3.3 is satisfied for X by double counting the sum of all weights $c(e)$ of edges e that are incident to a vertex in X. On the one hand,

$$\sum_{v \in X} c(v) \geq \sum_{v \in X} g(v) \ ,$$

since every vertex v in X satisfies $c(v) \geq g(v)$. On the other hand, it holds that $\mathrm{Inc}(X) \subseteq \mathrm{Inc}(N(X))$ and all vertices v in $N(X)$ satisfy $c(v) \leq f(v)$. Thus,

$$\sum_{v \in X} c(v) \leq \sum_{v \in N(X)} c(v) \leq \sum_{v \in N(X)} f(v) \ .$$

In total, Equation 3.3 holds for X.

Now, assume that there exists no function $c\colon E \to \mathbb{N} \cup \{0\}$ such that Equation 3.2 is satisfied for all vertices v. For a function $c\colon E \to \mathbb{N} \cup \{0\}$, let N_c be the number of vertices v with $g(v) \leq c(v)$. Moreover, we define $M_c = \min\{g(v) - c(v) \mid v \in V,\, g(v) > c(v)\}$. Let C be the set of all functions $c\colon E \to \mathbb{N} \cup \{0\}$ that maximize N_c under the condition that $c(v) \leq f(v)$ holds for all vertices v. Let $c \in C$ be a function that minimizes M_c among all functions in C. Clearly, it holds that $N_c < |A| + |B|$ and $M_c > 0$. Let u be a vertex with $g(u) - c(u) = M_c$, without loss of generality $u \in A$. The case $u \in B$ works analogous.

Consider a path $p = (v_0 = u, v_1, v_2, \ldots, v_k)$ for some k such that $c(\{v_i, v_{i+1}\}) > 0$ for all odd integers i with $0 < i < k$. Then, it must hold that $c(v_i) = f(v_i)$ for all odd i and $c(v_i) = g(v_i)$ for all even i. Otherwise, if $c(v_j) < f(v_j)$ for some odd j or $c(v_j) \neq g(v_j)$ for some even j, then define $c'\colon E \to \mathbb{N} \cup \{0\}$ with $c'(\{v_i, v_{i+1}\}) = c(\{v_i, v_{i+1}\}) + 1$ for all even i with $i < j$ and $c'(\{v_i, v_{i+1}\}) = c(\{v_i, v_{i+1}\}) - 1$ for all odd i with $i < j$ and otherwise $c'(e) = c(e)$. Then, c' satisfies $0 \leq c'(v) \leq f(v)$ for all vertices v. Moreover, if $M_c = 1$ then $N_{c'} > N_c$, and if $M_c \geq 2$ then $N_{c'} = N_c$ and $M_{c'} < M_c$, a contradiction in both cases.

Now, consider the set P of all paths $p = (v_0 = u, v_1, v_2, \ldots, v_k)$ such that $c(\{v_i, v_{i+1}\}) > 0$ for all odd i. Define X to be the set of all vertices $v \in A$ on a path in P. We double count the sum of weights $c(e)$ of edges e that are incident to a vertex in X. On the one hand,

$$\sum_{v \in X} c(v) = c(u) + \sum_{v \in X \setminus \{u\}} c(v) < \sum_{v \in X} g(v) \ ,$$

since for every vertex v in $X \setminus \{u\}$ it holds that $c(v) = g(v)$ and $c(u) < g(u)$. On the other hand,

$$\sum_{v \in X} c(v) = \sum_{v \in N(X)} f(v) ,$$

since for every vertex v in $N(X)$ it holds that $c(v) = f(v)$ and there exists no edge $e = \{\tilde{u}, \tilde{v}\}$ with $c(e) > 0$ and $\tilde{u} \in N(X)$ and $\tilde{v} \in A \setminus X$. Otherwise, there would exist a longer path in P. With both equations it follows that

$$\sum_{v \in X} g(v) > \sum_{v \in N(X)} f(v) .$$

□

Theorem 3.6 contains a theorem by Berge and Las Vergnas [BLV78] as a special case (if we consider bipartite graphs), which is stated in Theorem 3.7.

Theorem 3.7 ([BLV78]). *Let $G = (V, E)$ be a graph and let $f, g \colon V \to \mathbb{N} \cup \{0\}$ be functions with $f(v) > 0$ and $f(v) \geq g(v)$ and $f(v)$, $g(v)$ even for all $v \in V$. Then there exists a function $c \colon E \to \mathbb{N} \cup \{0\}$ with $g(v) \leq c(v) \leq f(v)$ for all vertices $v \in V$ if and only if*

$$\sum_{v \in X} g(v) \leq \sum_{v \in N(X)} f(v)$$

for all independent sets $X \subseteq V$.

Observe that Theorem 3.3 is a special case of Theorem 3.6 with $g(v) = 1$ for all vertices v. In this case, we can assume that $c(e) \in \{0, 1\}$ for all $e \in E$. Otherwise, we set $c'(e) = 1$ if $c(e) \geq 1$ and $c'(e) = 0$ if $c(e) = 0$. Then, c' satisfies Equation 3.2 if and only if c satisfies Equation 3.2 since $g(v) = 1$ for all vertices v.

For a function $f \colon V \to \mathbb{N}$, a graph $G = (V, E)$ is called *f-soluble* if there exists a function $c \colon E \to \mathbb{N} \cup \{0\}$ such that $f(v) = c(v)$ for all vertices $v \in V$. See [Tut52] for example. An *f-factor* is a set of edges $M \subseteq E$ such that such that $\deg_M(v) = f(v)$ for all vertices v. If we apply Theorem 3.6 with $f(v) = g(v)$ for all vertices $v \in V$, then we obtain an equivalence for f-solubility of bipartite graphs. If it additionally holds that $c(e) \in \{0, 1\}$ for all $v \in V$, then the set $M = \{e \in E \mid c(e) = 1\}$ is an f-factor. But from Theorem 3.6, we do not obtain an equivalence for the existence of f-factors in bipartite graphs.

It is possible to generalize Theorem 3.3 and Corollary 3.4 to arbitrary graphs. The method of alternating paths only works if $f(v) \geq 2$ for all vertices v. If $f(v) = 1$ for all vertices v, then we consider perfect matchings and here, we have an equivalence for the existence of perfect matchings by Tutte's Theorem. Here, we say that a component of a graph G is *odd* if it has an odd number of vertices. Moreover, for a set S of vertices, we define $G - S$ to be the induced graph $G[V \setminus S]$, i.e. the graph that is made of G by deleting all vertices in S and all edges that are incident to a vertex in S.

Theorem 3.8 (Tutte's Theorem [Tut47]). *A graph $G = (V, E)$ has a perfect matching if and only if for every set $S \subseteq V$, the size of S is at least the number of odd components in $G - S$.*

Theorem 3.9 ([BLV78]). *Let $G = (V, E)$ be a graph and let $f \colon V \to \mathbb{N}$ be a function with $f(v) \geq \min\{2, \deg(v)\}$ for all vertices $v \in V$. Then there exists an f-shallow hitting edge set in G if and only if*

$$|X| \leq \sum_{v \in N(X)} f(v)$$

holds for all independent sets X in G.

Proof. Let M be an f-shallow hitting edge set and X an independent set in G. Note that $X \cap N(X) = \emptyset$. We double count the set $\mathrm{Inc}(X) \cap M$. On the one hand, $|\mathrm{Inc}(X) \cap M| \geq |X|$ since every vertex in X is covered at least once by an edge in M and no two vertices in X are covered by the same edge in M. On the other hand, $|\mathrm{Inc}(X) \cap M| \leq \sum_{v \in N(X)} f(v)$ since every vertex in $N(X)$ is covered at most $f(v)$ times by an edge in M. In total, $|X| \leq \sum_{v \in N(X)} f(v)$.

Now, assume that G has no f-shallow hitting edge set. Let M be an f-shallow edge set that maximizes the number of covered vertices such that $|M|$ is minimal among all f-shallow edge sets that maximize the number of covered vertices. Let u be an uncovered vertex. Consider all alternating paths starting in u. Let $P = (u, v_1, \ldots, v_k)$ be such an alternating path. If $\deg_M(v_i) > 1$ for some even integer i or $\deg_M(v_i) < f(v_i)$ for some odd integer i, then we have an f-augmenting path. Thus, $\deg_M(v_i) = 1$ for all even integers i and $\deg_M(v_i) = f(v_i)$ for all odd integers i. Define X' to be the set of all vertices v such that there exists an alternating path $P = (u, v_1, \ldots, v_k)$ with $v = v_i$ for some even integer i. Define Y to be the set of all vertices v such that there exists an alternating path $P = (u, v_1, \ldots, v_k)$ with $v = v_i$ for some odd integer i. Let $X = X' \cup \{u\}$. Observe that $\deg_M(v) = 1$ for all vertices $v \in X'$ and $\deg_M(v) = f(v)$ for all vertices $v \in Y$. Moreover, observe that $X \cap Y = \emptyset$. For contradiction, assume that there exists a vertex v in $X \cap Y$, then $\deg_M(v) = 1$ and $\deg_M(v) = f(v)$. Thus, $f(v) = 1$ and $\deg(v) = 1$. Since $v \in X \cap Y$, there exists a path $P = (u, v_1, \ldots, v_{k-1}, v)$ with $\{v_{k-1}, v\} \in M$ and a path $P' = (u, v'_1, \ldots, v'_{k'-1}, v)$ with $\{v'_{k'-1}, v\} \notin M$. This is a contradiction to $\deg(v) = 1$.

For $v \in V$, let $N_M(v) = \{w \in V \mid \{v, w\} \in M\}$ and $N_M(Y) = \cup_{v \in Y} N_M(v)$. In the next step we show that $X \cap N(X) = \emptyset$ and $Y \cap N_M(Y) = \emptyset$. Assume that there exists a vertex $v \in X \cap N(X)$. Then, there exists an alternating path $P = (u, v_1, v_2, \ldots, v_{k-1}, v)$ with k even and an edge $e = \{v, w\} \in E$ with $w \in X$. Define M' to be the edge set that results from M by augmenting the path P and adding the edge e to M'. Then, for every vertex v_i on the path P it holds that $\deg_{M'}(v_i) = \deg_M(v_i)$ if $v_i \notin \{u, w\}$. For the vertex u it holds that $\deg_{M'}(u) = \deg_M(u) + 1 = 1$. For the vertex w it holds that $\deg_{M'}(w) = \deg_M(w) + 1 = 2$. Thus, M' is an f-shallow edge set covering more vertices than M, a contradiction. Now assume that there exists a vertex $v \in Y \cap N_M(Y)$. That is, there exists a vertex $w \in Y$ with $\{v, w\} \in M$. There exists an alternating path $P = (u, v_1, v_2, \ldots, v_{k-1}, v)$ with k odd. Observe that $\deg(v) \geq 2$. Moreover, $\deg(w) \geq 2$ since otherwise $w \in X$. Then, $\deg_M(v) \geq 2$ and $\deg_M(w) \geq 2$. Thus, we can remove the edge e from M and obtain an f-shallow edge set M' covering the same number of vertices with $|M'| < |M|$, a contradiction to the minimality of $|M|$.

Since $X \cap N(X) = \emptyset$ and we considered all alternating paths starting in u, we have $Y = N(X)$. Additionally, X is an independent set. Moreover, since $Y \cap N_M(Y) = \emptyset$ we have $X = N_M(Y)$. Let $M' \subseteq M$ be the set of all edges in M that have a vertex in X. We double count the set M'. On the one hand, $|M'| = |X| - 1$ since every vertex in X has degree 1 in M', except u that has degree 0 in M'. On the other hand, $|M'| = \sum_{v \in N(X)} f(v)$ since every vertex in $N(X)$ has degree $f(v)$ in M'. In total, we get

$$|X| > \sum_{v \in N(X)} f(v) \; .$$

□

Theorem 3.9 is a special case of a theorem by Berge and Las Vergnas [BLV78], which is stated in Theorem 3.10.

Theorem 3.10 ([BLV78])**.** *Let $G = (V, E)$ be a graph and let $f \colon V \to \mathbb{N}$ be a function with $f(v) \geq 1$ for all $v \in V$. If the subgraph induced by all vertices $v \in V$ with $f(v) = 1$ is bipartite or empty, then the following statements are equivalent.*

1. *G has an f-shallow hitting edge set.*
2. *$|X| \leq \sum_{v \in N(X)} f(v)$ for all independent sets $X \subseteq V$.*

By setting $f(v) = t$ for all vertices v in Theorem 3.9 for some integer $t \geq 2$, we obtain the following corollary.

Corollary 3.11 ([BLV78])**.** *Let $G = (V, E)$ be a graph and let $t \geq 2$ be an integer. Then there exists a t-shallow hitting edge set in G if and only if*

$$|X| \leq t|N(X)|$$

holds for all independent sets X in G.

By considering μ-near regular graphs, we obtain the following corollary.

Corollary 3.12. *Let $G = (V, E)$ be a μ-near regular graph. Then G has a* $\max\{2, \lceil \mu \rceil\}$*-shallow hitting edge set.*

Proof. Let $\delta = \delta(G)$ be the minimum degree and $\Delta = \Delta(G)$ be the maximum degree of G. Let X be an independent set in G. Thus it holds that $X \cap N(X) = \emptyset$. We count the number of incident edges to vertices in X in two ways. On the one hand, $|\operatorname{Inc}(X)| \geq \delta \cdot |X|$ since every vertex in X has at least δ incident edges. On the other hand, $|\operatorname{Inc}(X)| \leq \Delta \cdot |N(X)|$. Thus,

$$|X| \leq \frac{\Delta}{\delta} \cdot |N(X)| \leq \mu \cdot |N(X)| \leq \lceil \mu \rceil \cdot |N(X)| \ .$$

and by Corollary 3.11, G has a $\max\{2, \lceil \mu \rceil\}$-shallow hitting edge set. □

3.2 Hypergraphs of Large Minimum Degree

In the next corollary, we show a sufficient condition for the existence of t-shallow hitting edge sets in bipartite graphs of large minimum vertex degree and show that this condition in tight. It is well-known that every bipartite graph with parts of size n has a perfect matching if $\deg(v) \geq n/2$ for all vertices v. See [KO06] for example. We generalize this result to t-shallow hitting edge sets.

Corollary 3.13. *Let $G = (A \dot\cup B, E)$ be a bipartite graph with $|A| \leq |B| \leq t|A|$. Let δ_A respectively δ_B be the minimum degree of the vertices in A respectively B. If $t\delta_A + \delta_B \geq |A|$ and $\delta_A + t\delta_B \geq |B|$ then there exists a t-shallow hitting edge set.*

Proof. By Corollary 3.4, we have to show that $|X| \leq t|N(X)|$ holds for all sets $X \subseteq A$ and all sets $X \subseteq B$.

We show that every set $X \subseteq A$ satisfies $|X| \leq t|N(X)|$. First, assume that $1 \leq |X| \leq t\delta_A$. Since every vertex in X has at least δ_A neighbors, we have $|N(X)| \geq \delta_A$ and thus $|X| \leq t|N(X)|$. Now assume that $|X| > t\delta_A$. It follows that $|A \setminus X| < |A| - t\delta_A \leq \delta_B$. Since every vertex in B has at least δ_B neighbors in A, it must have a neighbor in X. Thus, $N(X) = B$ and $t|N(X)| = t|B| \geq t|A| \geq |X|$.

Now, we show that every set $X \subseteq B$ satisfies $|X| \leq t|N(X)|$. First, assume that $1 \leq |X| \leq t\delta_B$. Since every vertex in X has at least δ_B neighbors, we have $|N(X)| \geq \delta_B$ and thus $|X| \leq t|N(X)|$. Now assume that $|X| > t\delta_B$. It follows that $|B \setminus X| < |B| - t\delta_B \leq \delta_A$. Since every vertex in A has at least δ_A neighbors in A, it must have a neighbor in X. Thus, $N(X) = A$ and $t|N(X)| = t|A| \geq |B| \geq |X|$. □

As a special case of Corollary 3.13, assume that $|A| = |B| = n$. It follows that if G has minimum vertex degree $\delta(G) \geq n/(t+1)$ then G has a t-shallow hitting edge set. Indeed,

$$t\delta_A + \delta_B \geq \frac{tn}{t+1} + \frac{n}{t+1} = n = |A|$$

and

$$\delta_A + t\delta_B \geq \frac{n}{t+1} + \frac{tn}{t+1} = n = |B| \ .$$

To see that the condition $\delta(G) \geq n/(t+1)$ is tight, let $G' = (A \dot\cup B, E)$ be a bipartite graph with parts of size $|A| = |B| = n$ such that $t+1$ divides $n-1$. Let $A = A_1 \dot\cup A_2$ and $B = B_1 \dot\cup B_2$ where $|A_1| = |B_2| = (n-1)/(t+1)$ and $|A_2| = |B_1| = (nt+1)/(t+1)$ such that $|A| = |B| = n$. The edges of G' are exactly the pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ with $a_1 \in A_1, a_2 \in A_2, b_1 \in B_1, b_2 \in B_2$. The graph G' has minimum degree $\delta(G') = (n-1)/(t+1) \geq n/(t+1) - 1$ but does not contain a t-shallow hitting edge set as

$$t|N(B_1)| = t|A_1| = t \cdot \frac{n-1}{t+1} = \frac{tn+1}{t+1} - 1 = |B_1| - 1 < |B_1| \ .$$

We extend the definition of the minimum degree of a bipartite graph in the following way to m-uniform m-partite hypergraphs, see for example [KO06] and [AGS09]. Let $H = (V_1 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph. We say that a set $\hat{e} \subseteq V$ of vertices of H is *legal* if $|\hat{e} \cap V_i| \leq 1$ for all parts V_i of H. If $\hat{e} \subseteq V$ is a legal set of vertices of H with $|\hat{e}| = m-1$, the *neighborhood* $N_{m-1}(\hat{e})$ of $\hat{e}$ is the set of all vertices v such that $\hat{e} \cup \{v\}$ is an edge in H. Given an m-uniform m-partite hypergraph H, we define the *minimum degree* $\delta'_{m-1}(H)$ to be the minimum of $|N_{m-1}(\hat{e})|$ over all legal sets $\hat{e}$ of vertices in H with $|\hat{e}| = m-1$. The minimum degree $\delta'_{m-1}(H)$ is also called *minimum co-degree*. Note the difference of this definition to the definition of the minimum vertex-degree $\delta(H)$, which is the minimum number of incident edges of a vertex in H. We consider this definition of $\delta'_{m-1}(H)$ only in this section.

Kühn and Osthus [KO06] proved a sufficient bound on $\delta'_{m-1}(H)$ for the existence of perfect matchings in m-uniform m-partite hypergraphs. This bound was improved in [AGS09] and is stated in Theorem 3.14.

Theorem 3.14 ([AGS09])**.** *Let $m \geq 2$ and let $H = (V = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph with parts of size n such that for every legal $(m-1)$-set $\hat{e}$ contained in $V \setminus V_1$ we have $|N_{m-1}(\hat{e})| > n/2$ and for every legal $(m-1)$-set $\hat{f}$ contained in $V \setminus V_m$ we have $|N_{m-1}(\hat{f})| \geq n/2$. Then there exists a perfect matching in H.*

The upper bound in Theorem 3.14 is tight up to the condition $|N_{m-1}(\hat{e})| > n/2$ for every legal $(m-1)$-set $\hat{e}$ contained in $V \setminus V_1$. It is not known whether this condition can be replaced by $|N_{m-1}(\hat{e})| \geq n/2$. On the other hand, there exists an m-uniform m-partite hypergraph H with $\delta'_{m-1}(H) \geq n/2 - 1$ that has no perfect matching. This construction is due to [KO06] and shows that the result is tight.

In this section, we extend the upper and lower bound to t-shallow hitting edge sets in m-uniform m-partite hypergraphs H with large minimum degree $\delta'_{m-1}(H)$. By extending a construction from [KO06], we obtain the following construction of a hypergraph H with large minimum degree $\delta'_{m-1}(H)$ but without a t-shallow hitting edge set.

Theorem 3.15. *For all positive integers $m \geq 2$, $t \geq 2$ and n there exists an m-uniform m-partite hypergraph H with parts of size n and*

$$\delta'_{m-1}(H) \geq \frac{n}{(m-1)t+1} - 1$$

that has no t-shallow hitting edge set.

Proof. Let $\delta' = n/((m-1)t+1) - 1$. We define the m-partite m-uniform hypergraph $H = (V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m, E)$ with $|V_i| = n$ for all $i = 1, 2, \ldots, m$. Let V_i' be a subset of V_i of size $|V_i'| = \lceil \delta' \rceil$, for $i = 1, 2, \ldots, m$. Note that $|V_1' \cup V_2' \cup \cdots \cup V_m'| < m(\delta' + 1)$. We define that $e = \{v_1, v_2, \ldots, v_m\}$ is an edge in H if and only if $v_1 \in V_1, v_2 \in V_2, \ldots, v_m \in V_m$ and there is at least one v_i with $v_i \in V_i'$. Clearly, H has minimum degree $\delta'_{m-1}(H) \geq \delta'$. Assume, for contradiction, that there exists a t-shallow hitting edge set M of H. Then,

$$|M| \leq t|V_1' \cup V_2' \cup \cdots \cup V_m'| < tm(\delta' + 1)$$

since every vertex in $V_1' \cup V_2' \cup \cdots \cup V_m'$ is covered at most t times. On the other hand,

$$\begin{aligned} |M| &\geq \frac{1}{m-1} \left|(V_1 \setminus V_1') \cup (V_2 \setminus V_2') \cup \cdots \cup (V_m \setminus V_m')\right| > \frac{mn - m(\delta'+1)}{m-1} \\ &= \frac{mnt}{(m-1)t+1} = tm(\delta' + 1) \end{aligned}$$

since every vertex in $V_i \setminus V_i'$ is covered at least once and at most $m-1$ vertices of $(V_1 \setminus V_1') \cup \cdots \cup (V_m \setminus V_m')$ are covered by the same edge in M. Thus, $tm(\delta'+1) < |M| < tm(\delta'+1)$ is the desired contradiction. □

To obtain a sufficient condition, we first state a theorem from [KO06]. It says that if we relax the condition $\delta'_{m-1}(H) > n/2$ to $\delta'_{m-1}(H) \geq n/m$, then there exists an almost perfect matching, i.e. a matching that covers all but at most $m-2$ vertices from each part of H. In [KO06] it is shown that this result is tight as the minimum degree condition $\delta'_{m-1}(H) \geq n/m$ cannot be reduced.

Theorem 3.16 ([KO06])**.** *Let H be an m-uniform m-partite hypergraph with parts of size n and $\delta'_{m-1}(H) \geq n/m$. Then H has a matching that covers all but at most $m-2$ vertices in each part of H.*

In the next theorem, we generalize Theorem 3.14 to maximum t-shallow edge sets. Therefore, we first show in Lemma 3.19 a generalization of Theorem 3.16 to t-shallow edge sets that cover almost all vertices of each part. To show Lemma 3.19, we assume that there exists a graph with large minimum degree $\delta'_{m-1}(H)$ and show which properties such a graph has (Lemma 3.18). Then, we show that this lead to a contradiction in Lemma 3.19. Using Lemma 3.19 and raising the minimum vertex degree condition by 1, we can show that there exists a t-shallow hitting edge set in hypergraphs with large minimum vertex degree $\delta'_{m-1}(H)$.

In the first step, we show some properties of t-shallow hitting edge sets in m-uniform m-partite hypergraphs.

Lemma 3.17. *Let $m \geq 2$ and $t \geq 2$ be positive integers and $H = (V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph with parts of size n. Let M be a t-shallow edge set that satisfies the following properties. Here, let $n_{s,i}$ be the number of vertices $v \in V_i$ with* $\deg_M(v) = s$.

1. *There exists a non-negative integer n_0 with $n_{0,1} = n_{0,2} = \cdots = n_{0,m} = n_0$.*
2. $|M| \leq \lceil n/k \rceil (t-1) + n - n_0$.

Then, with $k = (m-1)t + 1$,

1. *$n_{t,i} \leq \lceil n/k \rceil$ for all $i = 1, 2, \ldots, m$.*
2. *If there exists a vertex $v \in V_i$ with $2 \leq \deg_M(v) < t$, then $n_{t,i} \leq \lceil n/k \rceil - 1$.*

3. *If* $|M| < \lceil n/k \rceil (t-1) + n - n_0$, *then* $n_{t,i} \leq \lceil n/k \rceil - 1$ *for all* $i = 1, 2, \ldots, m$.

4. *If* $|M| = \lceil n/k \rceil (t-1) + n - n_0$ *and* $|\{v \in e \mid \deg_M(v) \geq 2\}| \leq 1$ *for all* $e \in M$, *then* $n_{t,i} = \lceil n/k \rceil$ *for all* $i = 1, 2, \ldots, m$.

Proof. 1. For all $i = 1, 2, \ldots, m$, there exist $n - n_0 - n_{t,i}$ vertices in V_i that are covered at least once by M but less than t times. Thus, $|M| \geq t n_{t,i} + (n - n_0 - n_{t,i}) = (t-1)n_{t,i} + n - n_0$. By comparing this inequality with Property 2, we get $n_{t,i} \leq \lceil n/k \rceil$.

2. Using the same argument, we have $|M| \geq t n_{t,i} + (n - n_0 - n_{t,i}) + 1 = (t-1)n_{t,i} + n - n_0 + 1$. By comparing this inequality with Property 2, we get $n_{t,i} \leq \lceil n/k \rceil - 1/(t-1) < \lceil n/k \rceil$ and thus $n_{t,i} \leq \lceil n/k \rceil - 1$.

3. Using the same argument, we have $|M| \geq (t-1)n_{t,i} + n - n_0$. Since $|M| < \lceil n/k \rceil (t-1) + n - n_0$ it holds that $n_{t,i} < \lceil n/k \rceil$ and thus $n_{t,i} \leq \lceil n/k \rceil - 1$.

4. Let $n_s := n_{s,i}$ for all $s = 1, 2, \ldots, t$ and some arbitrary $i \in \{1, 2, \ldots, m\}$, and let $n_{\geq 2} = n_2 + n_3 + \cdots + n_t$. Our first goal is to upper-bound n_1 in the general case $n_t \leq \lceil n/k \rceil$ and in the case $n_t \leq \lceil n/k \rceil - 1$. For the general case, we upper-bound the size of M by $|M| \leq n_1 + t n_{\geq 2} = (t-1)n_{\geq 2} + n - n_0$ and with $|M| = \lceil n/k \rceil (t-1) + n - n_0$ it follows that $n_{\geq 2} \geq \lceil n/k \rceil$. Thus, we have $n_1 = n - n_0 - n_{\geq 2} \leq n - n_0 - \lceil n/k \rceil$.

   In the next step, we show that if $n_t \leq \lceil n/k \rceil - 1$ then $n_{\geq 2} \geq \lceil n/k \rceil + 1$. First note that $t \geq 3$ since for $t = 2$ it holds that $\lceil n/k \rceil + n - n_0 = |M| = n_1 + 2n_2 = n - n_0 + n_2 < \lceil n/k \rceil + n - n_0$, a contradiction. For $t \geq 3$, we have

$$\left\lceil \frac{n}{k} \right\rceil (t-1) + n - n_0 = |M| = n - n_0 + n_2 + 2n_3 + \cdots + (t-1)n_t$$
$$\leq n - n_0 + (t-2)(n_2 + n_3 + \cdots + n_{t-1}) + (t-1)n_t$$

   and thus

$$(t-1)\left(\left\lceil \frac{n}{k} \right\rceil - n_t\right) \leq (t-2)(n_2 + n_3 + \cdots + n_{t-1}) \ .$$

   It follows that

$$n_{\geq 2} \geq n_t + \frac{t-1}{t-2}\left(\left\lceil \frac{n}{k} \right\rceil - n_t\right) = \frac{1}{t-2}\left((t-1)\left\lceil \frac{n}{k} \right\rceil - n_t\right) > \left\lceil \frac{n}{k} \right\rceil$$

   and therefore $n_1 \leq n - n_0 - \lceil n/k \rceil - 1$ if $n_t \leq \lceil n/k \rceil - 1$.

   Now, we prove the claim. Assume, for contradiction, that there exists a part V_j with $n_{t,j} \leq \lceil n/k \rceil - 1$. Then, $n_{1,i} \leq n - n_0 - \lceil n/k \rceil$ for all $i = 1, 2, \ldots, m$ and $n_{1,j} = n - n_0 - (n_{2,j} + n_{3,j} + \cdots + n_{t,j}) \leq n - n_0 - \lceil n/k \rceil - 1$ since $n_{t,j} \leq \lceil n/k \rceil - 1$ and thus $n_{2,j} + n_{3,j} + \cdots + n_{t,j} \geq \lceil n/k \rceil + 1$. In the next step, observe that

$$(m-1)|M| \leq \sum_{i=1}^{m} n_{1,i} \ ,$$

   since each edge in M covers at least $(m-1)$ vertices that are covered once by M. This follows since each edge $e \in M$ has at most one vertex $v \in e$ with $\deg_M(v) \geq 2$. We bound the $n_{1,i}$'s in the sum and use $|M| = \lceil n/k \rceil (t-1) + n - n_0$ and obtain

$$(m-1)\left(\left\lceil \frac{n}{k} \right\rceil (t-1) + n - n_0\right) \leq \sum_{i=1}^{m} n_{1,i} \leq m\left(n - n_0 - \left\lceil \frac{n}{k} \right\rceil\right) - 1 \ .$$

   Simplifying this inequality and using $k = (m-1)t + 1$ and we get

$$\left\lceil \frac{n}{k} \right\rceil k + 1 \leq n - n_0 \ ,$$

a contradiction since $\lceil n/k \rceil \cdot k \geq n$ and $n_0 \geq 0$.

□

In the next step, we show some properties of t-shallow edge sets with specific properties that maximize the number of covered vertices. The intuition for the next lemma is, if the hypergraph H has large minimum vertex degree but no t-shallow edge set (with specific properties) that covers almost all vertices of each part, then H is not far from the hypergraph constructed in Theorem 3.15. In the following, we define $V(M)$ to be the union of all edges in M, given a hypergraph $H = (V, E)$ and a subset of edges M.

Lemma 3.18. *Let $m \geq 2$ and $t \geq 2$ be positive integers and $H = (V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph with parts of size n and minimum degree $\delta'_{m-1}(H) \geq \lceil n/k \rceil$ where $k = (m-1)t + 1$. Let M be a t-shallow edge set that satisfies the following properties and maximizes the number of covered vertices in H with respect to these properties. Here, let $n_{s,i}$ be the number of vertices $v \in V_i$ with $\deg_M(v) = s$.*

1. *There exists a non-negative integer n_0 such that $n_{0,1} = n_{0,2} = \cdots = n_{0,m} = n_0$.*
2. *$|M| \leq \lceil n/k \rceil (t-1) + n - n_0$.*

If $n_0 \geq m - 1$ then

1. *$N_{m-1}(\hat{e}) \subseteq V(M)$ for each legal $(m-1)$-set $\hat{e}$ of uncovered vertices of H.*
2. *$|M| = \lceil n/k \rceil (t-1) + n - n_0$.*
3. *For all edges $e \in M$ it holds that the number of vertices $v \in e$ that are covered at least twice by M is at most one, i.e. $|\{v \in e \mid \deg_M(v) \geq 2\}| \leq 1$ for all edges $e \in M$.*

Proof. We define $U_i = V_i \setminus V(M)$ to be the set of all uncovered vertices in V_i. Moreover, we define $U = U_1 \cup U_2 \cup \cdots \cup U_m$.

1. Let $\hat{e}$ be a legal $(m-1)$-set of uncovered vertices, that is $\hat{e} \subseteq U$. Assume that there exists a vertex $v \in U \cap N_{m-1}(\hat{e})$. Let $e = \{v\} \cup \hat{e}$. Then, the edge set $M' = M \cup \{e\}$ is t-shallow. Let $U'_i = V_i \setminus V(M')$ be the set of vertices of V_i that are not covered by M'. The edge set M' satisfies Property 1 with $n'_0 = |U'_1| = \cdots = |U'_m| = n_0 - 1$. Moreover, M' satisfies Property 2 since $|M'| = |M| + 1 = |M| + n_0 - n'_0 \leq \lceil n/k \rceil (t-1) + n - n'_0$. Since M' covers more vertices than M and satisfies all properties, we have the desired contradiction.

2. In the next step, we show that $|M| = \lceil n/k \rceil (t-1) + n - n_0$. Assume, for contradiction, that $|M| < \lceil n/k \rceil (t-1) + n - n_0$. By Lemma 3.17 Result 3, we have $n_{t,i} \leq \lceil n/k \rceil - 1$ for all parts V_i. Since $\delta'_{m-1}(H) \geq \lceil n/k \rceil$ and by Result 1 of this lemma, each legal $(m-1)$-set $\hat{e}$ of vertices has a neighboring vertex that is covered at least once but less than t times. For $i = 1, 2, \ldots, m$, we build an $(m-1)$-set $\hat{e}_i$ of vertices of $U \setminus U_i$, such that no two sets $\hat{e}_i$ contain the same vertex. This is possible since $n_0 = |U_i| \geq m - 1$ for all $i = 1, 2, \ldots, m$. Let $v_i \in N_{m-1}(\hat{e}_i)$ be a vertex with $1 \leq \deg_M(v_i) < t$. Observe that $v_i \in V_i$ for all $i = 1, 2, \ldots, m$. Define $e_i = \{v_i\} \cup \hat{e}_i$. We claim that

$$M' = M \cup \{e_i \mid i = 1, 2, \ldots, m\}$$

satisfies Property 1 and Property 2. Clearly, M' is t-shallow. Denote by $U'_i = V_i \setminus V(M')$ the set of vertices in V_i that are not covered by M'. Then, $|U'_i| = |U_i| - (m-1) = n_0 - m + 1$ since the edges in $\{e_j \mid j \neq i, j = 1, 2, \ldots, m\}$ cover $(m-1)$ vertices of U_i, but $e_i \cap U_i = \emptyset$. To show Property 2, note that $|M'| = |M| + m$ and $n'_0 := |U'_1| = \cdots = |U'_m| = n_0 - m + 1$. Thus, $|M'| = |M| + n_0 - n'_0 + 1 \leq \lceil n/k \rceil (t-1) + n - n'_0$.

3. We show that there exists no edge $e \in M$ such that at least two vertices of e are covered at least twice by M. Assume that there exists an edge $e = \{u_1, u_2, \dots, u_m\} \in M$, with $u_i \in V_i$ for $i = 1, 2, \dots, m$, such that at least two vertices of e are covered at least twice by M. Denote by $I \subseteq \{1, 2, \dots, m\}$ the set of indices such that $\{u_i \mid i \in I\}$ is the set of vertices of e that are covered at least twice by M. Then, $|I| \geq 2$. For each $i \in I$, we build a legal $(m-1)$-set $\hat{e}_i$ of vertices of $U \setminus U_i$, such that no two sets $\hat{e}_i$ contain the same vertex. This is possible since $n_0 = |U_i| \geq m - 1$ for all $i = 1, 2, \dots, m$ and $|I| \leq m$. By Result 1 of this lemma, $N_{m-1}(\hat{e}_i) \subseteq V_i \setminus U_i$. Denote by $W_i \subseteq V_i$ the set of vertices in V_i that are covered exactly t times by M. By Result 1 of Lemma 3.17, we have $|W_i| \leq \lceil n/k \rceil$ if $u_i \in W_i$ and, by Result 2 of Lemma 3.17, $|W_i| \leq \lceil n/k \rceil - 1$ if $u_i \notin W_i$, for $i \in I$. Since $|N_{m-1}(\hat{e}_i)| \geq \lceil n/k \rceil$, there exists a vertex $v_i \in N_{m-1}(\hat{e}_i)$ with $v_i \in (V_i \setminus (U_i \cup W_i)) \cup \{u_i\}$. That is, $1 \leq \deg_M(v_i) < t$ or $v_i = u_i$. Define $e_i = \{v_i\} \cup \hat{e}_i$. We claim that
$$M' = (M \setminus \{e\}) \cup \{e_i \mid i \in I\}$$
satisfies Property 1 and Property 2. Clearly, M' is t-shallow. Since $|I| \geq 2$ it holds that M' covers more vertices than M. Denote $U'_i = V_i \setminus V(M')$ to be the set of vertices in V_i that are not covered by M'. If $i \in I$, then there exist $|I| - 1$ edges in $\{e_j \mid j \in I\}$ that cover distinct vertices of U_i. Moreover, the edge e_i does not cover a vertex of U_i. Thus, $|U'_i| = |U_i| - (|I| - 1) = n_0 - |I| + 1$. If $i \notin I$, then all $|I|$ edges in $\{e_j \mid j \in I\}$ cover a vertex in U_i. But the vertex $v \in e \cap V_i$ is covered exactly once by M (by the definition of I) and thus, v is not covered by M'. Then, $|U'_i| = |U_i| - |I| + 1 = n_0 - |I| + 1$. To prove Property 2, note that $|M'| = |M| + |I| - 1$ and $n'_0 := |U'_1| = \cdots = |U'_m| = n_0 - |I| + 1$. Thus, $|M'| = |M| + (n_0 - n'_0) \leq \lceil n/k \rceil (t-1) + n - n_0 + n_0 - n'_0 = \lceil n/k \rceil (t-1) + n - n'_0$.

□

We can now prove that every m-uniform m-partite hypergraph H with minimum degree $\delta'_{m-1}(H) \geq n/((m-1)t+1)$ has a t-shallow edge set that covers all but at most $m-2$ vertices of each part. For $t = 1$, the result is proven by Theorem 3.16, so we consider the case $t \geq 2$. To deduce Theorem 3.20 from Lemma 3.19, we need to show some additional properties of the t-shallow edge set which we use in Theorem 3.20.

Lemma 3.19. *Let $m \geq 2$, $t \geq 2$ and n be positive integers and $H = (V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph with parts of size n and minimum degree $\delta'_{m-1}(H) \geq \lceil n/k \rceil$ where $k = (m-1)t + 1$. Then, there exists a t-shallow edge set M with the following properties. Here, let $n_{s,i}$ be the number of vertices $v \in V_i$ with $\deg_M(v) = s$.*

1. *There exists a non-negative integer n_0 with $n_{0,1} = n_{0,2} = \cdots = n_{0,m} = n_0$.*
2. *$|M| \leq \lceil n/k \rceil (t-1) + n - n_0$.*
3. *M covers all but at most $m-2$ vertices of each part of H, i.e. $n_0 \leq m-2$.*

Proof. Let M be a t-shallow edge set that satisfies Property 1 and Property 2 and maximizes the number of covered vertices with respect to these both properties. Assume, for contradiction, that $n_0 \geq m-1$. Then, we can apply Lemma 3.18. By Result 2 of Lemma 3.18, we have $|M| = \lceil n/k \rceil (t-1) + n - n_0$. By Result 3 of Lemma 3.18, every edge $e \in M$ has at most one vertex $v \in e$ with $\deg_M(v) \geq 2$. Then, by Result 4 of Lemma 3.17, we have $n_{t,i} = \lceil n/k \rceil$ for all $i = 1, 2, \dots, m$. We count the number of vertices in the part

V_1. Note that $n_{t,1} = \lceil n/k \rceil$. Since $n_{t,i} = \lceil n/k \rceil$ for all $i = 2, 3, \ldots, m$ and every edge in M has at most one vertex v with $\deg_M(v) \geq 2$, we have $n_{1,1} \geq t(m-1)\lceil n/k \rceil$. Thus,

$$n \geq n_0 + n_{1,1} + n_{t,1} \geq n_0 + t(m-1)\left\lceil \frac{n}{k} \right\rceil + \left\lceil \frac{n}{k} \right\rceil \geq n_0 + k\left\lceil \frac{n}{k} \right\rceil \geq n_0 + n \ ,$$

a contradiction since we assumed $n_0 \geq m - 1 > 0$. □

We are now able to prove that every m-uniform m-partite hypergraph with parts of size n and minimum degree $\delta'_{m-1}(H) \geq 1 + n/((m-1)t+1)$ has a t-shallow hitting edge set. This condition is tight up to the additive constant 1, as shown in Theorem 3.15.

Theorem 3.20. *Let $m \geq 2$ and $t \geq 2$ be positive integers and $H = (V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph with parts of size n and minimum degree*

$$\delta'_{m-1}(H) \geq \left\lceil \frac{n}{(m-1)t+1} \right\rceil + 1 \ .$$

Then, there exists a t-shallow hitting edge set in H.

Proof. Let $k = (m-1)t + 1$. By Lemma 3.19, there exists a t-shallow edge set M that satisfies Property 1, Property 2 and Property 3 of Lemma 3.19. Let M be such a t-shallow edge set that maximizes the number of covered vertices with respect to all three properties. Let $U_i = V_i \setminus V(M)$ be the set of uncovered vertices in V_i. Then, $|U_1| = \cdots = |U_m| = n_0$ for some non-negative integer $n_0 \leq m-2$, by Property 1 and Property 3. Moreover, $|M| \leq \lceil n/k \rceil(t-1) + n - n_0$ by Property 2. Let $n_{s,i}$ be the number of vertices $v \in V_i$ with $\deg_M(v) = s$. By Lemma 3.17 Result 1, we have $n_{t,i} \leq \lceil n/k \rceil$ for all $i = 1, 2, \ldots, m$. For $n_0 = 0$, there is nothing to show. Therefore, assume $n_0 > 0$. We construct a t-shallow hitting edge set M' by adding edges to M.

For each $i = 1, 2, \ldots, n_0+1$, we build a legal $(m-1)$-set $\hat{e}_i$ of vertices of $U \setminus U_i$, such that each vertex in $U_1 \cup U_2 \cup \cdots \cup U_{n_0+1}$ is contained in exactly one $\hat{e}_i$ and each vertex in $\hat{U}_{n_0+2} \cup \cdots \cup U_m$ is contained in at least one and at most two $\hat{e}_i$'s. Note that $N_{m-1}(\hat{e}_i) \subseteq V_i \setminus U_i$, since otherwise we can build a t-shallow edge set M' that covers more vertices and satisfies all three properties of Lemma 3.19, a contradiction. Since $\delta'_{m-1}(H) \geq \lceil n/k \rceil + 1$ and $n_{t,i} \leq \lceil n/k \rceil$ for all $i = 1, 2, \ldots, m$, there exists a vertex $v_i \in N_{m-1}(\hat{e}_i)$ that is covered at least once but less than t times by M. We define $e_i = \{v_i\} \cup \hat{e}_i$. Let $M' = M \cup \{e_i \mid i = 1, 2, \ldots, n_0 + 1\}$. Then, M' is t-shallow and covers all vertices of H. □

In the next two theorems, we study sufficient conditions on the minimum degree of uniform hypergraphs for the existence of t-shallow hitting edge sets. For that, we use the previous results for m-uniform m-partite hypergraphs. Let $H = (V, E)$ be an m-uniform hypergraph. If $\hat{e}$ is an $(m-1)$-set of vertices of H, the *neighborhood* $N_{m-1}(\hat{e})$ of $\hat{e}$ is the set of all vertices v such that $\hat{e} \cup \{v\}$ is an edge in H (analogous to m-uniform m-partite hypergraphs). We define the *minimum degree* $\delta_{m-1}(H)$ of an m-uniform hypergraph to be the minimum of $|N_{m-1}(\hat{e})|$ over all $(m-1)$-sets $\hat{e}$ of vertices in H. Note the difference to the definition of $\delta'_{m-1}(H')$ for an m-uniform m-partite hypergraph H', where we defined $\delta'_{m-1}(H')$ to be the minimum of $|N_{m-1}(\hat{e})|$ over all *legal* $(m-1)$-sets $\hat{e}$. Sometimes, both the minimum degree $\delta'_{m-1}(H')$ of an m-uniform m-partite hypergraph H' and the minimum degree $\delta_{m-1}(H)$ of an m-uniform hypergraph H are called *minimum co-degree* too. In the following, we clearly indicate when switching between $\delta'_{m-1}(H')$ and $\delta_{m-1}(H)$.

In the first theorem, we construct an m-uniform hypergraph H with large minimum degree $\delta_{m-1}(H)$ that has no t-shallow hitting edge set.

Theorem 3.21. *For all positive integers $m \geq 2$, $t \geq 2$ and n there exists an m-uniform hypergraph $H = (V, E)$ with $|V| = nm$ vertices and*

$$\delta_{m-1}(H) \geq \frac{|V|}{(m-1)t+1} - 1$$

that has no t-shallow hitting edge set.

Proof. Let $k = (m-1)t + 1$ and $\delta = |V|/k - 1$. We define an m-uniform hypergraph $H = (V, E)$ with $V = A \dot\cup B$ where

$$|A| = \lceil \delta \rceil \quad \text{and} \quad |B| = nm - \lceil \delta \rceil \ .$$

Note that $|V| = |A| + |B| = nm$ and $|A| < |V|/k$ and $|B| > (k-1)|V|/k$. We define that $e \subseteq V$ is an edge in H if and only if $|e| = m$ and e has a vertex in A, i.e. $e \cap A \neq \emptyset$. Clearly, H has minimum degree $\delta_{m-1}(H) \geq \delta$. Assume, for contradiction, that there exists a t-shallow hitting edge set M of H. Since every vertex in A is covered at most t times, we have

$$|M| \leq t|A| < \frac{tmn}{k} \ .$$

On the other hand,

$$|M| \geq \frac{|B|}{m-1} > \frac{(k-1)mn}{(m-1)k} = \frac{tmn}{k} \ ,$$

since every vertex in B is covered at least once and at most $m-1$ vertices of B are covered by the same edge in M. Thus, $tmn/k < |M| < tmn/k$ is the desired contradiction. $\square$

In Theorem 3.23, we show that every m-uniform hypergraph of large minimum degree $\delta_{m-1}(H)$ has a t-shallow hitting edge set. Here, the minimum degree condition matches the lower bound in Theorem 3.21 up to an error term of order $\mathcal{O}(m^2\sqrt{n \log n})$. To prove the theorem, we use the following definition and lemma from [KO06]. Let $H = (V, E)$ be an m-uniform hypergraph with $|V| = nm$ vertices and let $N \subseteq V$ be a set of vertices. Let $V = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m$ be a partition of V. Then, this partition *splits N fairly* if $|V_i| = n$ and

$$\left| |N \cap V_i| - \frac{|N|}{m} \right| \leq 2m\sqrt{n \log n}$$

for all $i = 1, 2, \ldots, m$.

Lemma 3.22 ([KO06])**.** *For each integer $m \geq 2$ there exists an integer $n_0 = n_0(m)$ such that for each m-uniform hypergraph $H = (V, E)$ with $|V| = nm$ vertices and $n \geq n_0$ there exists a partition $V = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m$ with $|V_i| = n$ for all $i = 1, 2, \ldots, m$ that splits all neighborhoods $N_{m-1}(\hat{e})$ of all $(m-1)$-sets $\hat{e} \subseteq V$ fairly.*

In the next theorem, we show a sufficient condition on the minimum degree $\delta_{m-1}(H)$ of an m-uniform hypergraph H for the existence of t-shallow hitting edge sets. Therefore, we reduce the m-uniform hypergraph H with large minimum degree $\delta_{m-1}(H)$ to an m-uniform m-partite subhypergraph $H' \subseteq H$ with large minimum degree $\delta'_{m-1}(H')$. Then, we can apply Theorem 3.20 to find a t-shallow hitting edge set in H', which is also a t-shallow hitting edge set in H.

Theorem 3.23. *Let $t \geq 2$ be an integer. For every integer $m \geq 3$ there exists an integer $n_0 = n_0(m)$ such that the following holds for all $n \geq n_0$. If $H = (V, E)$ is an m-uniform hypergraph with $|V| = nm$ vertices which satisfies*

$$\delta_{m-1}(H) \geq \frac{|V|}{(m-1)t+1} + 2m^2\sqrt{n \log n} + m \ ,$$

then H has a t-shallow hitting edge set.

Proof. Let $V = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m$ be a partition of V that splits all neighborhoods $N_{m-1}(\hat{e})$ of all $(m-1)$-sets $\hat{e}$ fairly. This partition exists due to Lemma 3.22. We define $H' = (V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_m, E')$ to be the m-uniform m-partite subhypergraph that consists of all legal edges in E, i.e. $e \in E'$ if and only if $e \in E$ and $|e \cap V_i| = 1$ for all $i = 1, 2, \ldots, m$. Observe that H' has parts of size n. If N is the neighborhood of an $(m-1)$-set $\hat{e} \subseteq V$ in H, then $|N| \geq \delta_{m-1}(H)$. Since the partition splits N fairly, we have

$$|N \cap V_i| \geq \frac{|N|}{m} - 2m\sqrt{n \log n}$$

for all parts V_i of the partition. Thus, for the minimum degree $\delta'_{m-1}(H')$ of the m-uniform m-partite hypergraph H' it holds that

$$\begin{aligned}\delta'_{m-1}(H') &\geq \frac{\delta_{m-1}(H)}{m} - 2m\sqrt{n \log n} \geq \frac{n}{(m-1)t+1} + 2m\sqrt{n \log n} + 1 - 2m\sqrt{n \log n} \\ &= \frac{n}{(m-1)t+1} + 1 \ .\end{aligned}$$

By Theorem 3.20, H' has a t-shallow hitting edge set and hence also H. $\square$

3.3 A Lower Bound for Regular Hypergraphs

In this section, we provide a lower bound for the least integer $t = t(m)$ such that every m-uniform m-partite regular hypergraph has a t-shallow hitting edge set.

Theorem 3.24. *Let $m \geq 2$ and $t \geq 2$ be positive integers. There exists an m-uniform m-partite t-regular hypergraph which has no $(t-1)$-shallow hitting edge set for $t = \lfloor (1 + \log_2 m)/2 \rfloor$.*

Proof. We prove the theorem by giving an explicit construction of a hypergraph $H = (V, E)$. Let

$$V = \binom{[2t]}{t}$$

with

$$|V| = \binom{2t}{t} \leq 4^t = 4^{\lfloor (1+\log_2 m)/2 \rfloor} \leq 2m$$

be the set of vertices. Each vertex v is a t-element subset of $[2t]$ that indicates the incident edges of v. For $i = 1, \ldots, 2t$, denote by e_i the set of vertices v with $i \in v$. Then, let $E = \{e_i \mid i = 1, \ldots, 2t\}$ be the edge set of H. Observe that every t-set of edges has a common vertex. Each edge e_i contains exactly

$$|e_i| = \binom{2t-1}{t-1} = \frac{1}{2}\binom{2t}{t} = \frac{1}{2}|V| \leq m$$

vertices. Thus, H is $|V|/2$-uniform. For a vertex $v \in V$, denote by $\overline{v} = [2t] \setminus v$ the complementary vertex in V. It holds that $\mathrm{Inc}(v) \cap \mathrm{Inc}(\overline{v}) = \emptyset$, i.e. no edge is incident to both v and $\overline{v}$. Thus, we can partition the set of vertices into $|V|/2$ parts, each of size 2. Therefore, H is $|V|/2$-partite. Moreover, each vertex has exactly t incident edges. Thus, H is t-regular. By adding additional vertices and extending each hyperedge to size m in such a way that H remains t-regular, we obtain an m-uniform m-partite t-regular hypergraph $\tilde{H}$.

Suppose that there exists a $(t-1)$-shallow hitting edge set $M \subseteq E$ in $\tilde{H}$. If $|M| < |E|/2$, then there exists a subset of edges $F \subseteq E$ of size $|F| = t$ with $F \cap M = \emptyset$. Then, the vertex

$v \in \cap_{e \in F} e$ has no incident edge that is contained in M. In this case, M is not a hitting edge set. On the other hand, if $|M| \geq |E|/2$, then there exists a subset $F \subseteq E$ of size $|F| = t$ with $F \subseteq M$. Then, all incident edges of the vertex $v \in \cap_{e \in F} e$ are contained in M. In this case, M is not $(t-1)$-shallow. $\square$

3.4 The Lovász Local Lemma

We use the Lovász Local Lemma for proving that every m-uniform m-partite regular hypergraph has a t-shallow hitting edge set for t depending only on m. In stating the Lovász Local Lemma, we follow the explanations from [Juk11]. Imagine that there is a set of bad events $A_1, \dots, A_n$ in a probability space. If each bad event only occurs with small probability and each bad event is dependent of a small amount of other bad events, the Lovász Local Lemma is a tool to prove that it is possible that no bad event occurs. An important characteristic of the Lovász Local Lemma is that it makes no assumption about the number n of bad events. This distinguishes it from other techniques like the union bound

$$\Pr[\overline{A}_1 \dots \overline{A}_n] = 1 - \Pr[A_1 \cup \dots \cup A_n] \geq 1 - \sum_{i=1}^{n} \Pr[A_i] \; .$$

An event A is *mutually independent* of a set $B = \{B_1, \dots, B_n\}$ of events, if

$$\Pr[A \mid D_1, \dots, D_k] = \Pr[A]$$

for all non-negative integers $k \leq n$, all subsets $C = \{C_1, \dots, C_k\} \subseteq B$ and all $D_i \in \{C_i, \overline{C_i}\}$, $i = 1, \dots, k$ with $\Pr[D_1 \dots D_k] > 0$.

Definition 3.25. *Let $A_1, \dots, A_n$ be events. A graph $G = (V, E)$ is a* dependency graph *of these events if $V = \{A_1, \dots, A_n\}$ and for all i, A_i is mutually independent of all events A_j with $i \neq j$ and $\{A_i, A_j\} \notin E$. The* degree of dependence *of the events $A_1, \dots, A_n$ is the smallest possible maximum degree of a dependency graph.*

With these definitions, we can state the Lovász Local Lemma.

Lemma 3.26 (Lovász Local Lemma [Spe77])**.** *Let $A_1, \dots, A_n$ be events with $\Pr[A_i] \leq p$, for all $i = 1, 2, \dots, n$, and let d be their degree of dependence. If $\mathrm{e}p(d+1) \leq 1$ then $\Pr[\overline{A}_1 \overline{A}_2 \dots \overline{A}_n] > 0$.*

In the first proof of the Lovász Local Lemma by Erdős and Lovász in [EL73], the condition $\mathrm{e}p(d+1) \leq 1$ was replaced by the stronger condition $4pd \leq 1$. Spencer generalized the Lovász Local Lemma and showed in [Spe77] that the implication remains true under the condition $\mathrm{e}p(d+1) \leq 1$ as stated in Lemma 3.26.

3.5 An Upper Bound for Regular Hypergraphs

In Lemma 3.27, we prove a sufficient condition for $t = t(m, \mu)$ such that every m-uniform μ-near regular hypergraph has a t-shallow hitting edge set. Note that the hypergraph does not have to be m-partite. We will derive a bound for the special case of m-uniform m-partite regular hypergraphs as corollary.

Lemma 3.27. *Let $m \geq 2$ and t be positive integers and let $\mu \geq 1$ be a real number. Let $H = (V, E)$ be an m-uniform μ-near regular hypergraph. If*

$$\frac{t!}{t+1} \left(\frac{1}{\mu m} \right)^{t+1} \geq \mathrm{e} m^2 \tag{3.4}$$

then H has a t-shallow hitting edge set.

Proof. Let $\Delta = \Delta(H)$ be the maximum degree and $\delta = \delta(H)$ be the minimum degree of the hypergraph H. We build an edge set $M \subseteq E$ out of the following random experiment: For each vertex $v \in V$, we pick an incident edge $e \in \mathrm{Inc}(v)$ uniformly at random and add it to the edge set M. Clearly, $\deg_M(v) \geq 1$ holds for all $v \in V$ with probability 1. Observe that for $t \geq \Delta$, M is t-shallow with probability 1. Thereby, assume that $t < \Delta$. We use the Lovász Local Lemma to prove that there exists an edge set M that also satisfies $\deg_M(v) \leq t$ for all vertices $v \in V$. For a set F of edges, denote by $V(F)$ the set of vertices $\bigcup_{e \in F} e$.

Define the set $\mathcal{F}$ to be the set of all edge sets $F \subseteq E$ of size $t+1$ such that there exists a vertex $v \in V$ with $F \subseteq \mathrm{Inc}(v)$, i.e.

$$\mathcal{F} = \{F \subseteq E \mid |F| = t+1, \exists\, v \in V \colon F \subseteq \mathrm{Inc}(v)\} \ .$$

For a set $F \in \mathcal{F}$, we denote the event that $F \subseteq M$ by A_F. These are the bad events used in the Lovász Local Lemma.

To apply the Lovász Local Lemma, we have to find a bound p such that $\Pr[A_F] \leq p$ holds for all $F \in \mathcal{F}$. For a vertex $v \in V$, we denote by P_v the random variable over the domain $\mathrm{Inc}(v)$ that describes which edge is picked at the vertex v. Moreover, for $e \in E$ and $v \in E$, we define $B_{e,v}$ to be the event that $P_v = e$ and B_e to be the event that $e \in M$. Since every vertex has degree at least δ, it holds that $\Pr[B_{e,v}] = \Pr[P_v = e] \leq 1/\delta$ for all $e \in E$ and $v \in e$. For an edge $e \in E$, we can now bound the probability for the event B_e by

$$\Pr[B_e] = \Pr\left[\bigcup_{v \in e} B_{e,v}\right] \leq \sum_{v \in e} \Pr[B_{e,v}] \leq \frac{m}{\delta} \ .$$

For $e \in E$ and $F \subseteq E$ with $e \notin F$, it holds that $\Pr[B_e \mid \cap_{e' \in F} B_{e'}] \leq \Pr[B_e]$ and hence $\Pr[B_e \cap (\cap_{e' \in F} B_{e'})] \leq \Pr[B_e] \cdot \Pr[\cap_{e' \in F} B_{e'}]$. Thus, for each $F \in \mathcal{F}$, the probability that the event A_F occurs can be bounded by

$$\Pr[A_F] = \Pr\left[\bigcap_{e \in F} B_e\right] \leq \prod_{e \in F} \Pr[B_e] \leq \left(\frac{m}{\delta}\right)^{t+1} \ .$$

We construct a dependency graph $G_{\mathrm{D}} = (V_{\mathrm{D}}, E_{\mathrm{D}})$ with $V_{\mathrm{D}} = \{A_F \mid F \in \mathcal{F}\}$. Two events A_F and $A_{F'}$ are adjacent in G_{D} if and only if $V(F) \cap V(F') \neq \emptyset$. Note that the event B_e is mutually independent of all events $B_{e'}$ with $e \cap e' = \emptyset$. Thus, each event A_F for $F \in \mathcal{F}$ is mutually independent of all non-adjacent events $A_{F'}$.

In the next step, we have to bound the degree of dependence of these events. For an edge $e \in E$ and a vertex $v \in e$, denote by $\mathcal{F}_{e,v} \subseteq \mathcal{F}$ the set of edge sets $F \in \mathcal{F}$ with $e \in F$ and $F \subseteq \mathrm{Inc}(v)$. We bound the size of the sets $\mathcal{F}_{e,v}$. The edge e is fixed to be in all sets $F \in \mathcal{F}_{e,v}$ and the vertex v has degree at most Δ. Since every edge set $F \in \mathcal{F}$ has size exactly $t+1$, there are $\binom{\Delta-1}{t}$ ways to choose the remaining t edges. Thus,

$$|\mathcal{F}_{e,v}| \leq \binom{\Delta - 1}{t} \quad \text{for all } e \in E \text{ and } v \in e \ .$$

Let $F \in \mathcal{F}$ be an arbitrary but fixed set of edges. We count the number of edge sets $F' \in \mathcal{F}$ with $V(F) \cap V(F') \neq \emptyset$. Let $F' \in \mathcal{F}$ be an arbitrary edge set with $A_{F'}$ adjacent to A_F in the dependency graph G_{D}. Since $V(F) \cap V(F') \neq \emptyset$, there must exist a vertex $\tilde{v}$ in $V(F) \cap V(F')$ with an edge e' in F' which is incident to $\tilde{v}$. By definition of $\mathcal{F}$, there is a vertex $v' \in V$ with $F \subseteq \mathrm{Inc}(v')$. Since $e' \in F'$, it holds that $v' \in e'$. Thus, F' is in the set

$\mathcal{F}_{e',v'}$. Since the set F itself fulfills all the described properties, it is also in a set $\mathcal{F}_{e,v}$ and we count it too. We can now bound the degree of dependence d by

$$d+1 \leq \max_{F\in\mathcal{F}} \left| \bigcup_{\tilde{v}\in V(F)} \bigcup_{e'\in \mathrm{Inc}(\tilde{v})} \bigcup_{v'\in e'} \mathcal{F}_{e',v'} \right| \leq (t+1)m \cdot \Delta \cdot m \cdot \binom{\Delta-1}{t} .$$

Since we assumed $\Delta - 1 \geq t$, it holds that

$$\binom{\Delta-1}{t} \leq \frac{(\Delta-1)^t}{t!} \leq \frac{\Delta^t}{t!}$$

and thus

$$d+1 \leq \frac{t+1}{t!} m^2 \Delta^{t+1} . \tag{3.5}$$

To apply the Lovász Local Lemma, we calculate $\mathrm{e}p(d+1)$. With Equation 3.4, it follows that

$$\mathrm{e}p(d+1) \leq \mathrm{e} \cdot \left(\frac{m}{\delta}\right)^{t+1} \cdot \frac{t+1}{t!} m^2 \Delta^{t+1} \leq \mathrm{e}m^2 \frac{t+1}{t!} (\mu m)^{t+1} \leq 1 .$$

By the Lovász Local Lemma, the probability that no event A_F with $F \in \mathcal{F}$ occurs is greater than zero. Thus, there exists an edge set $M \subseteq E$ such that every vertex has degree $1 \leq \deg_M(v) \leq t$ in M. Otherwise, there would exist a vertex $v \in V$ with $\deg_M(v) \geq t+1$ and there would be at least one event A_F with $F \subseteq \mathrm{Inc}(v) \cap M$ of size $t+1$ which occurred, a contradiction. □

We are now able to prove an asymptotic upper bound for the least integer $t = t(\mu, m)$ such that every m-partite μ-near regular hypergraph has a t-shallow hitting edge set.

Theorem 3.28. *Let $m \geq 2$ be a positive integer and let $\mu \geq 1$ be a real number. Then, every m-uniform μ-near regular hypergraph has a t-shallow hitting edge set with*

$$t = \mathrm{e}\mu m \cdot (1 + o(1)) ,$$

where the o-notation is respective $\mu m \to \infty$.

Proof. We define

$$x = \mathrm{e}\mu m , \quad y = \frac{\mathrm{e}^2}{\sqrt{2\pi}} \mu^2 m^4 \quad \text{and} \quad t = 1 + \left\lceil x y^{1/x} \right\rceil .$$

Observe that $y^{1/x} = 1 + o(1)$ where the o-notation is respective $x \to \infty$. Thus, $t = \mathrm{e}\mu m \cdot (1 + o(1))$ where the o-notation is respective $\mu m \to \infty$, as claimed.

Now, we use Lemma 3.27 and show that t satisfies Equation 3.4. We bound the left-hand side of Equation 3.4 using Stirling's Formula [Rob55] $n! \geq \sqrt{2\pi n}(n/\mathrm{e})^n$. In the second inequality, we use $\sqrt{t}/(t+1) \geq 1/t$ for $t \geq 3$. In the third inequality, we use $t \geq t-1$:

$$\begin{aligned} \frac{t!}{t+1}\left(\frac{1}{\mu m}\right)^{t+1} &\geq \left(\frac{t}{\mathrm{e}\mu m}\right)^t \frac{\sqrt{2\pi t}}{\mu m (t+1)} \geq \left(\frac{t}{\mathrm{e}\mu m}\right)^t \frac{\sqrt{2\pi}}{\mu m} \frac{1}{t} = \left(\frac{t}{\mathrm{e}\mu m}\right)^{t-1} \frac{\sqrt{2\pi}}{\mathrm{e}\mu^2 m^2} \\ &\geq \left(\frac{t-1}{\mathrm{e}\mu m}\right)^{t-1} \frac{\sqrt{2\pi}}{\mathrm{e}\mu^2 m^2} = \left(\frac{t-1}{x}\right)^{t-1} \frac{\mathrm{e}m^2}{y} . \end{aligned}$$

It holds that $y^{1/x} \geq 1$ and thus $t - 1 \geq x$. Therefore, with the definition of t, we have

$$\frac{t!}{t+1}\left(\frac{1}{\mu m}\right)^{t+1} \geq \left(\frac{t-1}{x}\right)^x \frac{\mathrm{e}m^2}{y} \geq \left(y^{1/x}\right)^x \frac{\mathrm{e}m^2}{y} = \mathrm{e}m^2 .$$

Thus, Equation 3.4 is satisfied for $t = \mathrm{e}\mu m(1 + o(1))$ and by Lemma 3.27, H has a t-shallow hitting edge set. □

Algorithm 3.1: Randomized Constructive Lovász Local Lemma

Input: set of mutually independent random variables $\mathcal{P}$, set of events $\mathcal{A}$ determined by $\mathcal{P}$

Output: Assignments of the random variables in $\mathcal{P}$ such that no event in $\mathcal{A}$ occurs.

forall $P \in \mathcal{P}$ **do**
  $v_P \leftarrow$ random evaluation of P;
while $\exists A \in \mathcal{A}$ such that A occurs when $P = v_P \; \forall P \in \mathcal{P}$ **do**
  pick an arbitrary event A that occurs;
  forall $P \in \mathrm{vbl}(A)$ **do**
    $v_P \leftarrow$ new random evaluation of P;
return $(v_P)_{P \in \mathcal{P}}$;

Note: Lemma 3.27 and Theorem 3.28 still hold if we consider μ-near regular hypergraphs where each edge contains at most m vertices. We need this relaxed version to prove Lemma 7.2 in Chapter 7.

By setting $\mu = 1$ in Theorem 3.28 and considering m-uniform m-partite regular hypergraphs, we obtain the following Corollary.

Corollary 3.29. *Every m-uniform m-partite regular hypergraph has a t-shallow hitting edge set with*

$$t = em \cdot (1 + o(1)) \; .$$

3.6 The Constructive Lovász Local Lemma

The Lovász Local Lemma, as stated is Lemma 3.26, is a non-constructive tool to prove the existence of an object. In [MT10], Moser and Tardos presented a constructive variant of the Lovász Local Lemma. In this section, we follow [MT10] in stating the Constructive Lovász Local Lemma in Theorem 3.30.

A set $\mathcal{P}$ of random variables *determines* an event A if, given any assignment of the random variables in $\mathcal{P}$, the probability of A is either 0 or 1. A set $\mathcal{P}$ of random variables *determines* a set of events $\mathcal{A}$ if $\mathcal{P}$ determines each event $A \in \mathcal{A}$. A finite set $\mathcal{P}$ of random variables is *mutually independent* if each random variable $P \in \mathcal{P}$ is mutually independent of the set $\mathcal{P} \setminus \{P\}$.

Suppose that there exists a finite set $\mathcal{P}$ of mutual independent random variables that determines the set of bad events $\mathcal{A}$. Denote by $\mathrm{vbl}(A)$ the unique minimal subset of $\mathcal{P}$ that determines A. Using this notation, we can construct a dependency graph $G_{\mathrm{D}} = (V_{\mathrm{D}}, E_{\mathrm{D}})$ with $V_{\mathrm{D}} = \mathcal{A}$ and two events A, A' in $\mathcal{A}$ adjacent if and only if $A \neq A'$ and $\mathrm{vbl}(A) \cap \mathrm{vbl}(A') \neq \emptyset$. Clearly, each event $A \in \mathcal{A}$ is mutually independent of all non-adjacent events A'.

The randomized algorithm for the Lovász Local Lemma (Algorithm 3.1) works as follows. For each $P \in \mathcal{P}$, we denote by v_P the current evaluation of the random variable P. In the first step, we pick a random evaluation for each $P \in \mathcal{P}$. While there exists a bad event $A \in \mathcal{A}$ that occurs, given the evaluation v_P for each $P \in \mathcal{P}$, the algorithm resamples the underlying events in $\mathrm{vbl}(A)$. This step is called *resampling* of A.

Moser and Tardos proved the following Constructive Lovász Local Lemma.

Lemma 3.30 (Constructive Lovász Local Lemma [MT10])**.** *Let $\mathcal{P}$ be a finite set of mutually independent random variables and let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a finite set of events determined by $\mathcal{P}$. If there exist real numbers $x_1, \dots, x_n$ with $0 < x_i < 1$, such that, for all i,*

$$\Pr[A_i] \le x_i \cdot \prod_{\{A_i, A_j\} \in E_D} (1 - x_j) \ ,$$

then there exists an assignment of values to the random variables in $\mathcal{P}$ such that no event in $\mathcal{A}$ occurs. Moreover, the randomized Algorithm 3.1 resamples each event $A_i \in \mathcal{A}$ at most an expected $x_i/(1 - x_i)$ times before finding such an evaluation.

3.7 A Constructive Upper Bound for Regular Hypergraphs

With the Constructive Lovász Local Lemma, we can prove that there exists a randomized algorithm that finds a $(\mathrm{e}\mu m(1+o_{\mu m}(1)))$-shallow hitting edge set in every m-uniform μ-near regular hypergraph in expected polynomial time. Here, the subscript in the o-notation denotes the quantity that tends to infinity.

Theorem 3.31. *Let $m \ge 2$ and t be positive integers. Let $H = (V, E)$ be an m-uniform hypergraph with maximum degree $\Delta = \Delta(H)$, minimum degree $\delta = \delta(H)$ and $n = |V|$. Then, there exists a randomized algorithm with expected runtime*

$$\mathcal{O}\left(n^2 \sqrt{\frac{\Delta\delta}{m}} \left(1 + \frac{m^2}{n} \cdot \frac{\log \Delta}{\delta}\right)\right)$$

that outputs a t-shallow hitting edge set with $t = \mathrm{e}\mu m(1 + o_{\mu m}(1))$.

Proof. We use the Constructive Lovász Local Lemma 3.30 to prove Theorem 3.31. We use the same random experiment as in the proof of Lemma 3.27 to build an edge set $M \subseteq E$ and, for each vertex $v \in V$, we denote by P_v the random variable over the domain $\mathrm{Inc}(v)$ that describes which edge is picked at the vertex v. Let $\mathcal{P}$ be the set of these random variables P_v, i.e. $\mathcal{P} = \{P_v \mid v \in V\}$. As in the proof of Lemma 3.27, let $\mathcal{F}$ be the set of all edge sets $F \subseteq E$ of size $t + 1$ such that there exists a vertex $v \in V$ with $F \subseteq \mathrm{Inc}(v)$. The set of bad events $\mathcal{A} = \{A_F \mid F \in \mathcal{F}\}$ is determined by $\mathcal{P}$.

For each event $A_F \in \mathcal{A}$, the probability $\Pr[A_F]$ is bounded by

$$\Pr[A_F] \le \left(\frac{m}{\delta}\right)^{t+1} \ .$$

For each event $A_F \in \mathcal{A}$, the set $\mathrm{vbl}(A)$ is the set of events $P_v \in \mathcal{P}$ with $v \in V(F)$. Thus, we can use the dependency graph $G_{\mathrm{D}} = (V_{\mathrm{D}}, E_{\mathrm{D}})$ constructed in the proof of Lemma 3.27. Then, the degree of dependence d can be bounded with the use of Equation 3.4 and Equation 3.5 by

$$d + 1 \le \frac{t+1}{t!} m^2 \Delta^{t+1} \le \frac{\Delta^{t+1}}{\mathrm{e}(\mu m)^{t+1}} = \frac{1}{\mathrm{e}} \left(\frac{\delta}{m}\right)^{t+1} =: d' + 1 \ .$$

To apply Lemma 3.30, we define x_F for $F \in \mathcal{F}$ to be $x_F = 1/(d' + 1)$. Since $\mathrm{e} \ge (1 + 1/d)^d$ for $d \ge 1$ it follows that $(1 - 1/(d + 1))^d \ge \mathrm{e}^{-1}$. Then, for each $F \in \mathcal{F}$,

$$x_F \prod_{\{A_F, A_{F'}\} \in E_{\mathrm{D}}} (1 - x_{F'}) = \frac{1}{d' + 1} \left(1 - \frac{1}{d' + 1}\right)^d \ge \frac{1}{(d' + 1)\mathrm{e}} = \left(\frac{m}{\delta}\right)^{t+1} \ge \Pr[A_F] \ .$$

By Lemma 3.30, the randomized Algorithm 3.1 finds an evaluation of $\mathcal{P}$ such that no event in $\mathcal{A}$ occurs. The initialization of the variables v_P runs in time $\mathcal{O}(n \log \Delta)$. To find a set F such that the event A_F occurs in the evaluation $(v_P)_{P\in\mathcal{P}}$ takes time $\mathcal{O}(n\Delta)$. Then, calculating new random evaluations v_P of the events $P \in \mathcal{P}$ takes time $\mathcal{O}(tm \log \Delta) = \mathcal{O}(\mu m^2 \log \Delta)$. Thus, one resampling step takes time $\mathcal{O}(n\Delta + \mu m^2 \log \Delta)$.

To calculate the total number of resampling steps, we need the size of the set $\mathcal{F}$. By the definition of $\mathcal{F}$, for each $F \in \mathcal{F}$, there exists a vertex $v \in V$ with $F \subseteq \mathrm{Inc}(v)$. There are n vertices and for each vertex $v \in V$, there are at most $\binom{\Delta}{t+1}$ sets $F \subseteq E$ with $F \subseteq \mathrm{Inc}(v)$. Thus, the size of $\mathcal{F}$ can be bounded by

$$|\mathcal{F}| \le n \cdot \binom{\Delta}{t+1} \le n \cdot \frac{\Delta^{t+1}}{(t+1)!}$$

Then, the expected number of resample steps is given by

$$\sum_{F\in\mathcal{F}} \frac{x_F}{1-x_F} \le \frac{|\mathcal{F}|/(d'+1)}{1-1/(d'+1)} = \frac{|\mathcal{F}|}{d'} = \frac{|\mathcal{F}|}{d'+1} \cdot \left(1+\frac{1}{d'}\right) .$$

With $d' \ge 1$ if follows that $1 + 1/d' \le 2$. Using the definition of d' and the bound for the size of $\mathcal{F}$, we obtain

$$\sum_{F\in\mathcal{F}} \frac{x_F}{1-x_F} \le 2n \cdot \frac{\Delta^{t+1}}{(t+1)!} \cdot \mathrm{e} \left(\frac{m}{\delta}\right)^{t+1} .$$

With $(t+1)! \ge \sqrt{2\pi t}(t/\mathrm{e})^{t+1}$ and $t \ge \mathrm{e}\mu m$, it follows that

$$\sum_{F\in\mathcal{F}} \frac{x_F}{1-x_F} \le \frac{2\mathrm{e}}{\sqrt{2\pi t}} n \left(\frac{\mathrm{e}m\mu}{t}\right)^{t+1} \le \frac{2\mathrm{e}}{\sqrt{2\pi t}} n \le \sqrt{\frac{2\mathrm{e}}{\pi}} \cdot \frac{n}{\sqrt{\mu m}} = \mathcal{O}\left(\frac{n}{\sqrt{\mu m}}\right)$$

is an upper bound to the expected total number of resampling steps. Thus, the expected running time of Algorithm 3.1 is

$$\mathcal{O}\left(\frac{n}{\sqrt{\mu m}} \cdot \left(n\Delta + \mu m^2 \log \Delta\right)\right) = \mathcal{O}\left(n^2 \sqrt{\frac{\Delta\delta}{m}} \left(1 + \frac{m^2}{n} \cdot \frac{\log \Delta}{\delta}\right)\right) .$$

□

4. Maximum Shallow Edge Sets

In this chapter, we consider the maximum size of shallow edge sets in m-uniform m-partite regular hypergraphs. In comparison to t-shallow hitting edge sets, we relax the condition $1 \leq \deg(v) \leq t$ to $\deg(v) \leq t$ for all vertices $v \in V$. Additionally, we are not interested in minimizing t for a given m (as for shallow hitting edge sets) but in maximizing the size of a t-shallow edge set for given t and m in m-uniform m-partite regular hypergraphs. We show that every m-uniform m-partite regular hypergraph H with n vertices per part has a t-shallow edge set of size

$$\Omega\left(\frac{nt}{m^{1/t}}\right) ,$$

where $t \leq \Delta(H)$. This generalizes the known bound for maximum matchings in m-uniform m-partite regular hypergraphs, which we summarize in Section 4.1. Moreover, we show in Section 4.3, that the lower bound above is tight.

For this, we use the following more formal framework. For a hypergraph H, let $\nu_t(H)$ be the size of a maximum t-shallow edge set in H. We define $\nu(H) = \nu_1(H)$ to be the size of a maximum matching in H. Let H be an m-uniform m-partite hypergraph with n vertices per part. Observe that $\nu_t(H) \leq nt$ is an upper bound on the size of a t-shallow edge set in H since every vertex in each part can be covered at most t times. If H is an m-uniform m-partite hypergraph with n vertices per part, we define by $\eta_t(H) = \nu_t(H)/(nt)$. We are particularly interested in bounding $\eta_t(H)$ for such hypergraphs H. For this, let $\mathcal{H}_t(m, \mu)$ be the set of all m-uniform m-partite μ-near regular hypergraphs with parts of equal sizes and $\Delta(H) \geq t$. We only consider such hypergraphs with $\Delta(H) \geq t$ since for $\Delta(H) < t$, the whole edge set is trivially a maximum t-shallow edge set. We define

$$\eta_t(m, \mu) = \inf_{H \in \mathcal{H}_t(m,\mu)} \eta_t(H)$$

Moreover, we define $\eta_t(m) = \eta_t(m, 1)$, $\eta(m, \mu) = \eta_1(m, \mu)$ and $\eta(m) = \eta_1(m)$. Observe that $0 \leq \eta_t(H) \leq 1$ for every hypergraph $H \in \mathcal{H}_t(m, \mu)$ and thus $0 \leq \eta_t(m, \mu) \leq 1$ for every positive integer m and every real number $\mu \geq 1$.

It is well-known that $m^{-1} \leq \eta(m) \leq (m-1)^{-1}$. The lower bound follows from Theorem 4.1. The upper bound follows from a construction explained in Section 5.1, namely the truncated projective plane. We show in Section 4.2 that

$$\eta_t(m, \mu) \geq \frac{1 - o_t(1)}{\mathrm{e}} \cdot \mu^{-1} m^{-1/t} .$$

This result is tight for $\mu = 1$ up to a constant factor. In fact, we show that for every real number $\epsilon > 0$ there exist infinitely many positive integers m such that

$$\eta_t(m) \leq (1 + \epsilon) \cdot m^{-1/t}$$

by providing a construction through combinatorial designs. Later, in Section 5.2, we will provide an explicit construction through projective spaces that shows a similar upper bound.

4.1 Maximum Matchings

It is well-known that every m-uniform m-partite regular hypergraph has a matching of size at least n/m, where n is the number of vertices per part. We show this result in Theorem 4.1. It follows that $\eta(m) \geq m^{-1}$.

Theorem 4.1. *Every m-uniform m-partite regular hypergraph H with n vertices per part has a matching of size at least n/m.*

Proof. First, observe that the minimum vertex cover in H has size exactly n. On the one hand, the vertices of one arbitrary part build a vertex cover of size n. On the other hand, if r is the regularity of the hypergraph, then every vertex covers r edges. Since there are rn edges in total, n vertices are required to cover all edges.

Given a maximum matching of size k, the union of the edges in the matching builds a vertex cover of size km. Therefore it must hold that $km \geq n$, which shows that there exists a matching of size at least n/m. □

4.2 A Lower Bound for Regular Hypergraphs

Lemma 4.2. *Let $m \geq 2$, t and k be positive integers. Let $H = (V_1 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph with maximum degree $\Delta = \Delta(H)$. If*

$$\frac{t!}{t+1} \left(\frac{k}{\Delta}\right)^t \geq \mathrm{e}m \tag{4.1}$$

then there exist k disjoint t-shallow edge sets $M_1, \ldots, M_k$ such that $M_1 \dot\cup \cdots \dot\cup M_k = E$.

Proof. We consider the following random experiment. For each edge $e \in E$, we pick a color $\chi(e) \in \{1, \ldots, k\}$ uniformly at random. The colors correspond to the sets M_i, $i = 1, \ldots, k$. We use the Lovász Local Lemma to prove that there exists an edge coloring such that no vertex has $t+1$ incident edges of the same color. For this, we define $\mathcal{F}$ to be the set of all edge sets $F \subseteq E$ of size $t+1$ such that there exists a vertex $v \in V$ with $F \subseteq \mathrm{Inc}(v)$, i.e.

$$\mathcal{F} = \{F \subseteq E \mid |F| = t+1, \exists v \in V \colon F \subseteq \mathrm{Inc}(v)\} \ .$$

For a set $F \in \mathcal{F}$, denote by A_F the event that all edges in F received the same color. Clearly,

$$\Pr[A_F] = \frac{1}{k^t} =: p \ .$$

We construct a dependency graph $G_{\mathrm{D}} = (V_{\mathrm{D}}, E_{\mathrm{D}})$ with $V_{\mathrm{D}} = \{A_F \mid F \in \mathcal{F}\}$ and two events A_F and $A_{F'}$ adjacent if and only if $F \cap F' \neq \emptyset$. Then, each event A_F is mutually independent of all non-adjacent events $A_{F'}$.

In the next step, we have to bound the degree of dependence d of the events A_F for $F \in \mathcal{F}$. For an edge $e \in E$ and a vertex $v \in V$, denote by $\mathcal{F}_{e,v}$ the set of all edge sets $F \in \mathcal{F}$ with

$e \in F$ and $F \subseteq \mathrm{Inc}(v)$. Note that the edge e is fixed to be in all sets $F \in \mathcal{F}_{e,v}$. Since $F \subseteq \mathrm{Inc}(v)$ for all $F \in \mathcal{F}_{e,v}$, there are at most $\binom{\Delta-1}{t}$ ways to choose the remaining edges. Hence, the size of each $\mathcal{F}_{e,v}$ can be bounded by

$$|\mathcal{F}_{e,v}| \leq \binom{\Delta - 1}{t} .$$

Let A_F be an arbitrary but fixed event with $F \in \mathcal{F}$. For every adjacent event $A_{F'}$, it holds that $F \cap F' \neq \emptyset$. Thus, there must be an edge e in $F \cap F'$. Moreover, by the definition of $\mathcal{F}$, there exists a vertex v such that $F \subseteq \mathrm{Inc}(v)$. Then, the degree of dependence d can be bounded by

$$d + 1 \leq \max_{F \in \mathcal{F}} \left| \bigcup_{e \in F} \bigcup_{v \in e} \mathcal{F}_{e,v} \right| \leq (t+1) \cdot m \cdot \binom{\Delta - 1}{t} \leq \frac{t+1}{t!} m \Delta^t .$$

To apply the Lovász Local Lemma, we calculate $\mathrm{e}p(d+1)$ and use Equation 4.1:

$$\mathrm{e}p(d+1) \leq \frac{\mathrm{e}}{k^t} \cdot \frac{t+1}{t!} m \Delta^t \leq 1 .$$

By the Lovász Local Lemma, the probability that no event A_F with $F \in \mathcal{F}$ occurs is greater than zero. Thus, there exists an edge coloring $\chi\colon E \to \{1, \dots, k\}$ such that no vertex has $t+1$ incident edges of the same color. For $i = 1, \dots, k$, we define M_i to be the set of all edges of color i. Then, each set M_i is t-shallow and it holds that $M_1 \dot\cup \cdots \dot\cup M_k = E$. $\square$

Lemma 4.3. *Let $m \geq 2$ and t be positive integers. Let $H = (V_1 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite hypergraph with maximum degree $\Delta = \Delta(H) \geq t$. Then, there exist k disjoint t-shallow edge sets $M_1, \dots, M_k$ such that $M_1 \dot\cup \cdots \dot\cup M_k = E$ for*

$$k = \frac{\mathrm{e}\Delta m^{1/t}}{t} \cdot (1 + o(1)) ,$$

where the o-notation is respective $t \to \infty$.

Proof. We prove the theorem by showing that

$$k = \left\lceil \frac{\mathrm{e}\Delta}{t} \left(\frac{t+1}{\sqrt{2\pi t}} \mathrm{e} m \right)^{1/t} \right\rceil$$

satisfies Equation 4.1. For that, we use Stirling's Formula $n! \geq \sqrt{2\pi n}(n/e)^n$ and the definition of k:

$$\left(\frac{k}{\Delta}\right)^t \frac{t!}{t+1} \geq \left(\frac{tk}{\Delta \mathrm{e}}\right)^t \frac{\sqrt{2\pi t}}{t+1} \geq \frac{t+1}{\sqrt{2\pi t}} \mathrm{e}m \cdot \frac{\sqrt{2\pi t}}{t+1} \geq \mathrm{e}m .$$

By Lemma 4.2, there exist k disjoint t-shallow edge sets $M_1, \dots, M_k$ with $M_1 \dot\cup \cdots \dot\cup M_k = E$. With

$$\left(\frac{t+1}{\sqrt{2\pi t}} \mathrm{e} \right)^{1/t} = 1 + o(1)$$

it holds that

$$k = \frac{\mathrm{e}\Delta m^{1/t}}{t} \cdot (1 + o(1)) ,$$

where the o-notation is respective $t \to \infty$. $\square$

Theorem 4.4. *Let n, t and $m \geq 2$ be positive integers and $\mu \geq 1$ a real number. Let $H = (V_1 \dot\cup \cdots \dot\cup V_m, E)$ be an m-uniform m-partite μ-near regular hypergraph with n vertices per part and maximum degree $\Delta = \Delta(H) \geq t$. Then there exists a t-shallow edge set of size at least*

$$\frac{nt}{\mathrm{e}\mu m^{1/t}} \cdot (1 - o(1)) \ ,$$

where the o-notation is respective $t \to \infty$.

Proof. Let $\Delta = \Delta(H)$ be the maximum degree and $\delta = \delta(H)$ be the minimum degree of H. By Lemma 4.3, for

$$k = \frac{\mathrm{e}\Delta m^{1/t}}{t} \cdot (1 + o_t(1))$$

there exist k disjoint t-shallow edge sets $M_1, \dots, M_k$ with $M_1 \dot\cup \cdots \dot\cup M_k = E$. It holds that

$$|E| = |M_1| + \cdots + |M_k| \geq n\delta \ .$$

By the pigeonhole principle, there exists an edge set M_i of size

$$\max_{i \in [k]} |M_i| \geq \frac{n\delta}{k} = \frac{n\delta t}{\mathrm{e}\Delta m^{1/t}} \frac{1}{1 + o_t(1)} = \frac{nt}{\mathrm{e}\mu m^{1/t}} \cdot (1 - o_t(1)) \ .$$

□

With Theorem 4.4, we obtain the following lower bound for $\eta_t(m, \mu)$.

Corollary 4.5. *Let t and $m \geq 2$ be positive integers and $\mu \geq 1$ a real number. Then,*

$$\eta_t(m, \mu) \geq \frac{1 - o_t(1)}{\mathrm{e}} \cdot \mu^{-1} m^{-1/t} \ .$$

In the next corollary, we consider the problem of covering all vertices by shallow edge sets. For a given positive integer s, the question is how many s-shallow edge sets do we need to cover all vertices of a given m-uniform m-partite μ-near regular hypergraph? Clearly, if $s \geq t(m, \mu)$ for $t(m, \mu)$ as defined in Chapter 3, we only need one s-shallow edge set since there exists an s-shallow hitting edge set. In Corollary 4.6 we give an upper bound for this question for all values of $s \geq 1$.

Corollary 4.6. *Let s and $m \geq 2$ be positive integers and $\mu \geq 1$ a real number. Let $H = (V, E)$ be an m-uniform m-partite μ-near regular hypergraph. Then there exist k disjoint s-shallow edge sets such that their union is a hitting edge set, where*

$$k = \max\left\{1\,,\ \frac{\mathrm{e}^2 \mu m^{1+1/s}}{s} \cdot (1 + o_s(1))(1 + o_{\mu m}(1))\right\} \ ,$$

where the subscript in the o-notation denotes the quantity that tends to infinity.

Proof. By Theorem 3.28, there exists a t-shallow hitting edge set M where

$$t = \mathrm{e}\mu m \cdot (1 + o_{\mu m}(1)) \ .$$

Then, the hypergraph $\tilde{H} = (V, M)$ is m-uniform, m-partite and has maximum degree $\Delta = \Delta(\tilde{H}) \leq t$. If $s \geq \Delta$, then M is already s-shallow. Thereby, assume that $s < \Delta$. Then, by Lemma 4.3, there exist k disjoint s-shallow edge sets $M = M_1 \dot\cup \cdots \dot\cup M_k$ where

$$k = \frac{\mathrm{e}\Delta m^{1/s}}{s} \cdot (1 + o_s(1)) \leq \frac{\mathrm{e}tm^{1/s}}{s} \cdot (1 + o_s(1)) = \frac{\mathrm{e}^2 \mu m^{1+1/s}}{s} \cdot (1 + o_s(1))(1 + o_{\mu m}(1)) \ .$$

□

By setting $s = 1$ in Corollary 4.6, we obtain the following corollary.

Corollary 4.7. *Let H be an m-uniform m-partite μ-near regular hypergraph. Then, there exist k matchings such that their union is a hitting edge set, where*

$$k = \mathcal{O}\left(\mu m^2\right) \ .$$

4.3 Combinatorial Designs

In this section, we show that there exists an m-uniform m-partite regular hypergraph that has only t-shallow edge sets of small size. This result follows from the existence of combinatorial designs, proved by Keevash [Kee18]. In Chapter 5, we will provide a simple construction with a similar upper bound.

Definition 4.8. *Let t, v, k and λ be positive integers. A set of* points *V with a multiset $\mathcal{B}$ of subsets of V (called* blocks*) is called a t-(v, k, λ)-*design *if*

1. *$|V| = v$ and*
2. *$|B| = k$ for each block $B \in \mathcal{B}$ and*
3. *for each set $U \subseteq V$ of size t there exist exactly λ blocks $B \in \mathcal{B}$ with $U \subseteq B$.*

First, we state a well-known property of combinatorial designs. This theorem gives a necessary condition for the existence of combinatorial designs.

Theorem 4.9 ([CD07])**.** *Let $(V, \mathcal{B})$ be a t-(v, k, λ)-design. If $I \subseteq V$ is a set of size $0 \leq |I| = i \leq t$, then the number of blocks containing I is*

$$r_i = \lambda \binom{v-i}{t-i} \Big/ \binom{k-i}{t-i} \ .$$

Proof. Let $I \subseteq V$ be a set of size i with $0 \leq i \leq t$. We double count the number N of pairs (U, B) with $I \subseteq U \subseteq B \in \mathcal{B}$ and $|U| = t$. On the one hand, $N = \lambda\binom{v-i}{t-i}$ since there exist $\binom{v-i}{t-i}$ sets U of size t with $I \subseteq U$ and each such U is contained in exactly λ blocks $B \in \mathcal{B}$. On the other hand, $N = r_i\binom{k-i}{t-i}$ since there are r_i blocks $B \in \mathcal{B}$ with $I \subseteq B$ and for each such block B there are $\binom{k-i}{t-i}$ sets U of size t with $I \subseteq U$. Thus, $N = \lambda\binom{v-i}{t-i} = r_i\binom{k-i}{t-i}$ and we obtain $r_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$. □

By this theorem, we obtain a necessary condition for the existence of a t-(v, k, λ)-design: for all integers i with $0 \leq i \leq t-1$ it must hold that $\binom{k-i}{t-i}$ divides $\lambda\binom{v-i}{t-i}$. Keevash [Kee18] showed in a recent, so far unpublished paper that this condition is also sufficient for large enough v. Here, a t-(v, k, λ)-design $(V, \mathcal{B})$ is called *resolvable* if the set $\mathcal{B}$ of blocks can be partitioned into perfect matchings.

Theorem 4.10 ([Kee18])**.** *Suppose $k \geq t \geq 1$ and λ are fixed and $v > v_0(k, t, \lambda)$ is sufficiently large such that $k \mid v$ and $\binom{k-i}{t-i} \mid \lambda\binom{v-i}{t-i}$ for all integers i with $0 \leq i \leq t-1$. Then there exists a resolvable t-(v, k, λ)-design.*

It immediately follows from Theorem 4.10 that for any positive integers t and k with $k \geq t \geq 1$ there exists a t-$(nk, k, 1)$-design for some $n > n_0(k, t)$.

Corollary 4.11. *Suppose $k \geq t \geq 1$ are fixed. Then there exist infinitely many positive integers n such that there exists a t-$(nk, k, 1)$-design.*

Proof. We use Theorem 4.10 to prove this corollary. Clearly, $k \mid nk$ and thus, the first condition is satisfied. Therefore, it suffices to show that there exist infinitely many positive integers n which satisfy $\binom{k-i}{t-i} \mid \binom{nk-i}{t-i}$ for all integers i with $0 \leq i \leq t-1$.

We claim that for all positive integers μ,

$$n = 1 + \mu \cdot k(k-1)(k-2)\cdots(k-t+1)$$

satisfies the divisibility conditions. Indeed, for all integers i with $0 \leq i \leq t-1$,

$$\frac{nk-i}{k-i} = n + \frac{n-1}{k-i} i$$

is a positive integer and therefore,

$$\frac{\binom{nk-i}{t-i}}{\binom{k-i}{t-i}} = \frac{(nk-i)(nk-i-1)\cdots(nk-t+1)}{(k-i)(k-i-1)\cdots(k-t+1)}$$

is a positive integer for all integers $i = 0, 1, \ldots, t-1$. □

Theorem 4.12. *Let t and k be positive integers with $k > t \geq 1$. Then there exist infinitely many positive integers n such that there exists an m-uniform m-partite k-regular hypergraph H with parts of size n and maximum t-shallow edge set of size t, where $m = \binom{nk-1}{t} / \binom{k-1}{t}$. This shows*

$$\eta_t(m) \leq \eta_t(H) \leq \frac{1}{1 - t/k} \cdot m^{-1/t}$$

for the specified values of m.

Proof. Let $H = (V, \mathcal{B})$ be a resolvable $(t+1)$-$(nk, k, 1)$-design. This exists due to Corollary 4.11 for infinitely many positive integers n. That is, $|V| = nk$, each block $B \in \mathcal{B}$ has size $|B| = k$ and each set of $t+1$ elements of V is contained in exactly one block in $\mathcal{B}$. Then, the hypergraph H is k-uniform, m-regular (because of Theorem 4.9) and has nk vertices and nm edges. Define $H^* = (V^*, E^*)$ to be the dual hypergraph of H. Then, H^* has nm vertices and nk edges. Moreover, H^* is m-uniform and k-regular. Since the design $H = (V, \mathcal{B})$ is resolvable, the hypergraph H^* is m-partite with parts of size n. Note that every set of $t+1$ edges in H^* has a vertex that is incident to all $t+1$ edges. Thus, the maximum t-shallow edge set has size t, i.e. $\nu_t(H) = t$. With

$$m = \frac{\binom{nk-1}{t}}{\binom{k-1}{t}} = \frac{(nk-1)(nk-2)\cdots(nk-t)}{(k-1)(k-2)\cdots(k-t)} \leq \left(\frac{nk}{k-t}\right)^t$$

we obtain $n \geq m^{1/t}(1 - t/k)$ and therefore

$$\eta_t(m) \leq \eta_t(H) = \frac{t}{tn} = \frac{1}{n} \leq \frac{1}{1 - t/k} \cdot m^{-1/t} \; .$$

□

For a fixed positive integer t, k can be chosen arbitrarily large. Therefore, the following corollary follows immediate.

Corollary 4.13. *Let t be a fixed positive integer. For every real number $\epsilon > 0$ there exist infinitely many positive integers m such that*

$$\eta_t(m) \leq (1+\epsilon)m^{-1/t} \ .$$

Proof. We choose k such that $t/k \leq \epsilon/(1+\epsilon)$ and apply Theorem 4.12. □

Thus, this upper bound asymptotically matches the lower bound obtained in Section 4.2 up to a constant factor.

5. A Generalization of the Truncated Projective Plane

In this chapter, we use a construction to prove an upper bound on the maximum size of t-shallow edge sets in m-uniform m-partite regular hypergraphs. Recall from Chapter 4, that $\nu_t(H)$ is the size of a maximum t-shallow edge set in H. Moreover, we defined $\eta_t(H) = \nu_t(H)/(nt)$ for an m-uniform m-partite hypergraph H with n vertices per part. In this chapter, we show that there exist such hypergraphs H such that

$$\nu_t(H) \leq \frac{tn}{m^{1/t} - 1}$$

is an upper bound of the maximum size of t-shallow edge sets, where n is the number of vertices per part. This shows

$$\eta_t(m) \leq (m^{1/t} - 1)^{-1}$$

for infinitely many values of m, where $\eta_t(m)$ is as defined in Chapter 4. Moreover, we use the same construction to improve the lower bound, stated in Section 3.3, on the least integer $t = t(m)$ such that every m-uniform m-partite regular hypergraph has a t-shallow hitting edge set. We show that

$$t(m) \geq \lfloor \log_2(m+1) \rfloor$$

is a lower bound.

Our construction is a generalization of the truncated projective plane. The (truncated) projective plane is introduced in Section 5.1. A generalization of projective planes are projective spaces, which are introduced in Section 5.2. In the same section, we construct truncated projective spaces which are a generalization of the truncated projective plane. Moreover, we use this construction to prove the bounds mentioned above.

5.1 The Truncated Projective Plane

In this section, we introduce the (truncated) projective plane in terms of hypergraphs. For this, we follow the explanations from [Kåh02].

Definition 5.1. *Let $H = (V, E)$ be a hypergraph. Then, H is a* projective plane *if it satisfies*

1. *each two distinct edges $e_1 \neq e_2$ have a unique vertex in common, i.e. $|e_1 \cap e_2| = 1$,*
2. *for each two distinct vertices $v_1 \neq v_2$ there exists a unique edge e that contains both v_1 and v_2,*
3. *there exist four vertices such that no edge contains three of them.*

Lemma 5.2 ([Kåh02])**.** *If H is a projective plane, then the dual hypergraph H^* is a projective plane.*

Proof. We have to prove that H^* satisfies all three properties of Definition 5.1. Since each two distinct edges in H have a unique vertex in common, each two distinct vertices in H^* have a unique edge that contains both vertices. Since each two distinct vertices of H have a unique edge that contains both, each two distinct edges in H^* have a unique vertex in common. Thus, Properties 1 and 2 hold for H^*.

In H, there exist four vertices such that no edge contains three of them. Thus, in H^*, there exist four edges e_1, e_2, e_3 and e_4 such that no vertex is incident to three of them. By Property 1, the unique vertices $v_{i,j}$ in H^* with $v_{i,j} \in e_i \cap e_j$, $1 \leq i < j \leq 4$, must be distinct. We claim that the four vertices $v_{1,2}, v_{2,3}, v_{3,4}$ and $v_{1,4}$ satisfy Property 3. Assume, for contradiction, that there exists an edge e containing three of them. This contradicts Property 1 of H^*, because we found two distinct edges in H^* that have at least two vertices in common. □

Lemma 5.3 ([Kåh02])**.** *Let $H = (V, E)$ be a projective plane. Then, there exists a positive integer $q > 1$ such that each edge contains $q+1$ vertices and each vertex is incident to $q+1$ edges. The positive integer q is called the* order *of the projective plane H.*

Proof. We first prove the following claim: For every edge e and every vertex $v \notin e$ it holds that $|e| = |\operatorname{Inc}(v)|$. Let e be an edge and $v \notin e$ a vertex in H. By Property 2, for every vertex u of e there exists a unique edge that contains both u and v. Moreover, by Property 1, e is the only edge that contains at least two vertices of e. Thus, $|e| \geq |\operatorname{Inc}(v)|$. On the other hand, for every edge $\tilde{e}$ incident to v, the edges e and $\tilde{e}$ have a unique vertex u in common (Property 1). By Property 2, there exist no two edges incident to both u and v. Thus, $|e| \leq |\operatorname{Inc}(v)|$.

By Property 3, there exist four vertices v_1, v_2, v_3 and v_4 such that no edge contains three of them. Let $e_{i,j}$, $1 \leq i < j \leq 4$, be the unique edge that contains both v_i and v_j. Observe that every vertex v_i, $1 \leq i \leq 4$, has at least three incident edges. For $q > 1$, let $q+1$ be the number of incident edges of v_1. By the previous claim, $|e_{2,3}| = |e_{3,4}| = |e_{2,4}| = q+1$. Let v be an arbitrary vertex in H. Then, at least two edges of $e_{2,3}, e_{3,4}, e_{2,4}$ do not contain v. By the claim, the number of incident edges of v is $q+1$. Thus, every vertex in H has the same number $q+1$ of incident edges.

Let e be an edge of H. Then, at least two vertices of v_1, v_2, v_3, v_4 are not contained in e. Thus, by the claim, the edge e has size $q+1$. □

Definition 5.4. *Let $H = (V, E)$ be a projective plane of order q and let $u \in V$ be an arbitrary vertex. Then, $H' = (V \setminus \{u\}, E')$ with $E' = \{e \in E \mid u \notin e\}$ is called* truncated projective plane of order *q.*

Lemma 5.5. *Let $H' = (V', E')$ be a truncated projective plane of order q. Then, H' is a $(q+1)$-uniform $(q+1)$-partite q-regular hypergraph with q vertices per part.*

Proof. Let $H = (V' \cup \{u\}, E)$ be the corresponding projective plane of order q. Let $\text{Inc}(u) = \{e_1, e_2, \ldots, e_{q+1}\}$ the set of incident edges of u in H. Since $|e_i \cap e_j| = 1$ for $i \neq j$, it holds that $e_i \cap e_j = \{u\}$. By Property 2, every vertex in V' is incident to an edge e_i for some i. Thus, we can partition the set of vertices into $q+1$ parts $V_i = e_i \setminus \{u\}$, $1 \leq i \leq q+1$. Since $|e_i| = q+1$, every part V_i has size q. For every vertex v in H, the number of incident edges of v in H is $q+1$. Then, for $v \neq u$, the number of incident edges of v in H' is q.

By Property 2, every edge in H has exactly one vertex in each part. Thus, H' is $(q+1)$-uniform $(q+1)$-partite. □

Lemma 5.6 ([Kåh02]). *Let $H = (V, E)$ be a projective plane of order q. Then, $|V| = |E| = q^2 + q + 1$.*

Proof. By Lemma 5.5, the corresponding truncated projective plane is $(q+1)$-uniform $(q+1)$-partite q-regular with q vertices per part. Thus, the number of vertices in H is $|V| = q(q+1) + 1 = q^2 + q + 1$ since we removed one vertex to obtain the truncated projective plane. The number of edges in H is $|E| = q^2 + (q+1)$ since we removed $q+1$ edges to obtain the truncated projective plane. □

The truncated projective plane is an extremal construction for Ryser's Conjecture, which is stated in Conjecture 5.7. For a hypergraph H, let $\tau(H)$ be the size of a minimum vertex cover of H and let $\nu(H)$ be the size of a maximum matching in H.

Let H be an m-uniform hypergraph. If M is a matching in H of size $\nu(H)$, then $V' = \cup_{e \in M} e$ is a vertex cover in H of size $m \cdot \nu(H)$. Assume, for contradiction that there exists an edge e in H with $e \cap V' = \emptyset$. Then, $M \cup \{e\}$ is a larger matching, a contradiction. Thus, $\tau(H) \leq m \cdot \nu(H)$. If H is additionally m-partite, then Ryser's Conjecture says that the statement can be strengthened to $\tau(H) \leq (m-1) \cdot \nu(H)$.

Conjecture 5.7 (Ryser's Conjecture). *For every m-uniform m-partite hypergraph H, $\tau(H) \leq (m-1) \cdot \nu(H)$.*

In the case $m = 2$, Ryser's Conjecture is proven by König's Theorem that says that in every bipartite graph, the size of a minimum vertex cover is equal to the size of a maximum matching. In the case $m = 3$, Ryser's Conjecture was solved by Aharoni in [Aha01]. For $m \geq 4$, Ryser's Conjecture is still open.

If H is a truncated projective plane of order q, then $\nu(H) = 1$ since each two edges have a vertex in common. Moreover, $\tau(H) = q$ since all q vertices of one part build a vertex cover and each vertex covers exactly q edges out of q^2 edges in total. Thus, it holds that $\tau(H) = q \cdot \nu(H)$. Since H is $(q+1)$-uniform $(q+1)$-partite, the truncated projective plane H satisfies Ryser's Conjecture with equality. Since H is m-uniform m-partite with $m = q+1$ and has parts of size $n = q$, the largest matching in H has size $n/(m-1)$. Note that this is nearly optimal since by Theorem 4.1, every m-uniform m-partite regular hypergraph with n vertices per part has a matching of size at least n/m. Motivated by this construction, we generalize the truncated projective plane to obtain similar results for shallow edge sets.

5.2 Projective Spaces

A generalization of projective planes are projective spaces. A projective plane is a 2-dimensional projective space. The vertices of a projective plane are also called *points* and the edges are called *lines*. Points are of dimension 0 and lines are of dimension 1. This concept can be generalized to arbitrary dimensions $d \geq 2$. For example, a projective space of dimension 3 has additionally *planes* with dimension 2. In introducing projective spaces, we follow the explanations from [Cas06].

Definition 5.8 ([Cas06])**.** *For an integer $d \geq -1$, a set S is a d-*dimensional projective space *if it satisfies the following axioms:*

1. *S is a set which elements are called* points.
2. *For every integer h with $-1 \leq h \leq d$ there exists a subset $U \subseteq S$ of dimension h. We say that U is a subspace of dimension h and write $\dim(U) = h$.*
3. *There exists a unique subspace of dimension -1.*
4. *The sets $P \subseteq S$ with $|P| = 1$ are exactly the subspaces of dimension 0.*
5. *The set S is the unique subspace of dimension d.*
6. *If U and V are subspaces with $U \subseteq V$, then $\dim(U) \leq \dim(V)$. Moreover, $U = V$ if and only if $\dim(U) = \dim(V)$ and $U \subseteq V$.*
7. *If U and V are subspaces, then the intersection $U \cap V$ is a subspace of S.*
8. *If U and V are subspaces, then the* span *of U and V, denoted by $\langle U, V\rangle$, is the intersection of all subspaces containing both U and V. It holds that*
$$\dim(\langle U, V\rangle) = \dim(U) + \dim(V) - \dim(U \cap V) \ .$$
9. *Every subspace of dimension 1 contains at least three points.*

Observe that, by these axioms, $\emptyset$ is the unique subspace of dimension -1. Moreover, note that by Axiom 7 and Axiom 8, the span $\langle U, V\rangle$ of two subspaces U and V is a subspace. The subspaces of dimension 0 are called *points*. The subspaces of dimension 1 are called *lines*. The subspaces of dimension 2 are called *planes*. The subspaces of dimension $d-1$ are called *hyperplanes*. By $\langle U_1, U_2, \ldots, U_k\rangle$ for some k, we denote the intersection of all subspaces containing all of $U_1, U_2, \ldots, U_k$. Thus,
$$\langle U_1, U_2, \ldots, U_k\rangle = \langle\langle \ldots \langle\langle U_1, U_2\rangle, U_3\rangle \ldots\rangle, U_k\rangle \ .$$

An important fact is that the subspaces of a projective space are projective spaces too. We prove this in the next lemma.

Lemma 5.9 ([Cas06])**.** *Let S be a d-dimensional projective space and T a subspace of S of dimension r. Then, T is an r-dimensional projective space.*

Proof. Let S be a d-dimensional projective space and T a subspace of S of dimension r. We define the subspaces of T to be the subspaces of S that are contained in T. If $r = -1$ then $T = \emptyset$ and T fulfills all axioms. If $r = 0$ then $|T| = 1$ and T fulfills all axioms. If $r = 1$ then T is a line with $|T| \geq 3$ (Axiom 9). It is easy to check that T fulfills all axioms.

Let T be a subspace of dimension $r \geq 2$. Obviously, T satisfies Axiom 1, 3, 4 and 9. In the next step, we prove Axiom 2 for T by induction on the dimension h. For $h = -1$, by Axiom 3 for T, there exists a subspace of dimension -1 that is contained in T. Suppose that there exists a subspace U of dimension $h-1$ in T for some h with $0 \leq h \leq r$. We prove that there must exist a subspace of dimension h. There must exist a point $P \subseteq T \setminus U$, otherwise we would have $T = U$ (Axiom 6 for S) and therefore $h - 1 = r$, a contradiction to

$h \leq r$. Define $U' = \langle P, U \rangle$. By Axiom 8 for S, we have $\dim(U') = 0 + (h-1) - (-1) = h$. We have to show that U' is completely contained in T. By Axiom 7 for S, $U'' = T \cap U'$ is a subspace of S. Then it holds that $U \subsetneq U'' \subseteq U'$ with $\dim(U) + 1 = \dim(U')$. By Axiom 6 for S, it must hold that $U' = U''$ and thus, U' is contained in T.

Assume that there exists an r-dimensional subset $T' \subseteq T$ in S, then by Axiom 6 for S it holds that $T = T'$. Thus, Axiom 5 holds for T. Axiom 6 and Axiom 7 hold for T because they hold for S and every subspace of T is a subspace of S.

Next, we prove Axiom 8 for T. Let U and V be two subspaces of S with $U, V \subseteq T$. We have to show that the span of U and V respective S, denoted by $\langle U, V \rangle_S$, is equal to the span of U and V respective T, denoted by $\langle U, V \rangle_T$. Obviously, $\langle U, V \rangle_S \subseteq \langle U, V \rangle_T$ since every subspace in T containing both U and V is a subspace in S containing both U and V. On the other hand, if A is a subspace in S containing both U and V, then $A' = T \cap A$ is a subspace in S containing both U and V. Since $A' \subseteq A$ and $A' \subseteq T$ it holds that A' is a subspace in T containing both U and V. Thus, since $A \cap A' = A'$ it holds that $\langle U, V \rangle_T \subseteq \langle U, V \rangle_S$. Overall, $\langle U, V \rangle_S = \langle U, V \rangle_T$. Since Axiom 8 holds for S it also holds for T. □

Lemma 5.10 ([Cas06]). *Let S be a 2-dimensional projective space and E be the set of lines in S. Then, the hypergraph $H = (S, E)$ is a projective plane.*

Proof. We need to prove the properties in Definition 5.1. Let l_1 and l_2 be two distinct lines in S. Then,

$$\dim(l_1 \cap l_2) = \dim(l_1) + \dim(l_2) - \dim(\langle l_1, l_2 \rangle) = 1 + 1 - 2 = 0 \ .$$

Hence, $|l_1 \cap l_2| = 1$ and there is a unique point in $l_1 \cap l_2$. Thus, Property 1 is satisfied.

Let P_1 and P_2 be two distinct points in S. Then $l = \langle P_1, P_2 \rangle$ is the unique line that contains P_1 and P_2. Thus, Property 2 is satisfied.

By Axiom 9 of S, each line in S has at least three points. Let A and B be points on a line $l = \langle A, B \rangle$ in S. There must exist a point $X \notin l$, otherwise S would have dimension 1. Since each line contains at least three points, there exist points $C \in \langle A, X \rangle$ and $D \in \langle B, X \rangle$, both distinct from A, B and X. Then, A, B, C and D are four distinct points such that no line contains three of them. Thus, Property 3 is satisfied. □

Lemma 5.11 ([Cas06]). *Let S be a projective space of dimension d. Let U be a hyperplane in S and let P and Q be points in S with $P \subseteq S \setminus U$ and $P \neq Q$. Then, $l = \langle P, Q \rangle$ is a line with $|l \cap U| = 1$, i.e. l intersects U in a unique point.*

Proof. We first prove that $l = \langle P, Q \rangle$ is a line. By Axiom 8, we have

$$\dim(l) = \dim(P) + \dim(Q) - \dim(\emptyset) = 0 + 0 - (-1) = 1 \ .$$

By the definition of the span, we have

$$\langle P, U \rangle \subseteq \langle P, Q, U \rangle = \langle l, U \rangle \subseteq S \ .$$

By Axiom 8, we have

$$\dim(\langle P, U \rangle) = \dim(P) + \dim(U) - \dim(\emptyset) = 0 + (d-1) - (-1) = d$$

and thus $\langle P, U \rangle = S$ and therefore $\langle l, U \rangle = S$. It follows that

$$\dim(l \cap U) = \dim(l) + \dim(U) - \dim(\langle l, U \rangle) = 1 + (d-1) - d = 0 \ .$$

Thus, $|l \cap U| = 1$. □

Lemma 5.12 ([Cas06])**.** *Let S be a d-dimensional projective space. Then there exists an integer $q > 1$ such that every r-dimensional subspace has size $N_r = q^r + q^{r-1} + \cdots + q + 1$ and is contained in N_{d-1-r} hyperplanes. The integer q is called the* order *of the projective space.*

Proof. We first prove the statement that there exists an integer $q > 1$ such that every r-dimensional subspace has size $N_r = q^r + q^{r-1} + \cdots + q + 1$. Observe that this statement is obviously true when $d \in \{-1, 0, 1\}$ or $r \in \{-1, 0\}$. In the first step we show that there exists an integer $q > 1$ such that every line in S has size $q + 1$. Let l and m be two distinct lines in S. If $\dim(\langle l, m\rangle) = 2$ then the subspace $\langle l, m\rangle$ is a projective plane. By Lemma 5.3, there exists an integer $q > 1$ such that l and m have size $q + 1$. If $\dim(\langle l, m\rangle) = 3$, then there exists a point $P \subseteq l$ and a point $Q \subseteq m$ with $P \neq Q$. Consider the line $k = \langle P, Q\rangle$. Then, $\dim(\langle l, k\rangle) = \dim(\langle m, k\rangle) = 2$. By the first case, $|l| = |k| = |m| = q + 1$ for some integer $q > 1$. Overall, there exists an integer $q > 1$ such that each line in S has size $q + 1$.

We prove the first statement of this lemma by induction over r. Suppose that every t-dimensional subspace of S has size $N_t = q^t + q^{t-1} + \cdots + q + 1$ for $-1 \leq t \leq r - 1$ and some $r \geq 2$. Let U be an r-dimensional subspace of S. Since U is a projective space of dimension r, there exists a hyperplane V in U, i.e. $V \subseteq U$ and $\dim(V) = r - 1$, and a point $P \subseteq U \setminus V$. Then, it holds that $U = \langle P, V\rangle$. By Lemma 5.11, each line in U incident to P intersects V in a unique point. Moreover, each line contains exactly $q + 1$ points and for each point Q in U there is exactly one line that contains P and Q. Thus, the size of U is exactly $N_r = qN_{r-1} + 1 = q^r + q^{r-1} + \cdots + q + 1$.

In the next step, we prove that every r-dimensional subspace is contained in N_{d-1-r} hyperplanes by induction over d. If $d \in \{-1, 0, 1\}$, the statement is obviously true. If $d = 2$, then S is a projective plane and the statement is true by Lemma 5.3. Let $d > 2$ and S be a d-dimensional projective space of order q. Let U be an arbitrary subspace of S. If $\dim(U) \geq d - 1$, then the statement is obviously true. Thus, let $\dim(U) \leq d - 2$. Then, there exists a $(d - 1)$-dimensional subspace $T \subseteq S$ that contains U. The number of points in $S \setminus T$ is

$$|S \setminus T| = \left(q^d + q^{d-1} + \cdots + q + 1\right) - \left(q^{d-1} + q^{d-2} + \cdots + q + 1\right) = q^d \ .$$

By the induction hypothesis, U is a subspace in T that is contained in $N_{(d-1)-1-r}$ hyperplanes of T. If V is a $(d - 2)$-dimensional hyperplane in T that contains U, then $\langle P, V\rangle$ is a $(d - 1)$-dimensional hyperplane in S that contains U if P is a point with $P \subseteq S \setminus T$. Since $|V| = q^{d-2} + \cdots + q + 1$ and $|\langle P, V\rangle| = q^{d-1} + \cdots + q + 1$, there are q^{d-1} points in $\langle P, V\rangle \setminus V$. Thus, there are $q^d/q^{d-1} = q$ distinct hyperplanes $\langle P, V\rangle$ with $P \subseteq S \setminus T$. Moreover, T itself is a hyperplane in S that contains U. Thus, the number of hyperplanes in S that contains U is at least $N_{d-1-r} = qN_{(d-1)-1-r} + 1$. On the other hand, if V is a hyperplane in S that contains U, then either $V = T$ or $V \cap T$ is a hyperplane in T that contains U. Thus, the number of hyperplanes in S that contains U is exactly N_{d-1-r}. □

Lemma 5.13 ([Cas06])**.** *Let S be a d-dimensional projective space of order q. Let S^* be the set of hyperplanes in S. If $U \subseteq S$ is a subspace of S, then $U^* = \{V \mid V$ hyperplane in S and $U \subseteq V\}$ is defined to be a subspace in S^* of dimension* $\dim(U^*) = d - 1 - \dim(U)$. *Then, S^* is a d-dimensional projective space of order q, called the* dual projective space *of S.*

Proof. To prove this theorem, we need to check the axioms in Definition 5.8. Obviously, the Axioms 1, 2, 3, 4, 5 are satisfied for S^*.

We prove that Axiom 6 holds for S^*. Let U^* and V^* be two subspaces of S^* with $U^* \subseteq V^*$. That is, for every subspace P^* in S^* it holds that $P^* \subseteq V^*$ if $P^* \subseteq U^*$. This is equivalent to the following proposition: For every subspace P in S it holds that $V \subseteq P$ if $U \subseteq P$. It follows that $V \subseteq U$. Assume, for contradiction, that there exists a point $Q \subseteq V \setminus U$. Then, there exists a subspace in S, namely U, that contains U but not Q, a contradiction. By Axiom 6 for S it holds that $\dim(V) \leq \dim(U)$ with equality if and only if $U = V$. Then, $\dim(U^*) \leq \dim(V^*)$ with equality if and only if $U^* = V^*$.

In the next step, we prove Axiom 7. Assume, for contradiction, that U^* and V^* are two subspaces in S^* but $U^* \cap V^*$ is not a subspace in S^*. Let W^* be a subspace with maximum dimension in S^* that is contained in $U^* \cap V^*$ and $P^* \subseteq (U^* \cap V^*) \setminus W^*$ a point in S^*. Then, P and W are two subspaces of S such that both contain U and V and moreover, $W \nsubseteq P$. Then, $P \cap W$ is a subspace of S with strictly smaller dimension than W that contains both U and V. Then, $(P \cap W)^*$ is a subspace of S^* with strictly higher dimension than W^* that is contained in $U \cap V$, a contradiction.

To prove Axiom 8, we prove the following claim. Let U and V be two subspaces of S, then

$$(U \cap V)^* = \langle U^*, V^* \rangle \tag{5.1}$$

and

$$\langle U, V \rangle^* = U^* \cap V^* \ . \tag{5.2}$$

First, we prove Equation 5.1. Since U, V and $U \cap V$ are subspaces of S, it holds that U^*, V^* and $(U \cap V)^*$ are subspaces of S^* with $U^*, V^* \subseteq (U \cap V)^*$. Let $W^* = \langle U^*, V^* \rangle$. By the definition of the span, it holds that $U^*, V^* \subseteq W^* \subseteq (U \cap V)^*$. Thus, it holds that $U \cap V \subseteq W \subseteq U, V$, which is equivalent to $W = U \cap V$. It follows that $(U \cap V)^* = \langle U^*, V^* \rangle$.

Next, we prove Equation 5.2. Since U, V and $\langle U, V \rangle$ are subspaces of S, it holds that U^*, V^* and $\langle U, V \rangle^*$ are subspaces of S^* with $\langle U, V \rangle^* \subseteq U^* \cap V^*$. Let $W^* = U^* \cap V^*$. It holds that $\langle U, V \rangle^* \subseteq W^* \subseteq U^*, V^*$. Thus, it holds that $U, V \subseteq W \subseteq \langle U, V \rangle$. Since the span $\langle U, V \rangle$ is the minimal subspace that contains U and V, it holds that $W = \langle U, V \rangle$. It follows that $\langle U, V \rangle^* = U^* \cap V^*$.

With Equations 5.1 and 5.2, we have

$$\begin{aligned}
\dim(\langle U^*, V^* \rangle) &= \dim((U \cap V)^*) = d - 1 - \dim(U \cap V) \\
&= d - 1 - \dim(U) - \dim(V) + \dim(\langle U, V \rangle) \\
&= d - 1 - \dim(U) - \dim(V) + d - 1 - \dim(U^* \cap V^*) \\
&= \dim(U^*) + \dim(V^*) - \dim(U^* \cap V^*)
\end{aligned}$$

which proves Axiom 8.

In the last step, we prove that the order of the projective space is preserved. By Lemma 5.12, every $(d-1-r)$-dimensional subspace of S is contained in $N_r = q^r + \cdots + q + 1$ hyperplanes of S. It follows that every r-dimensional subspace of S^* has size N_r. Moreover, every $(d-1-r)$-dimensional subspace of S has size N_{d-1-r}. It follows that every r-dimensional subspace of S^* is contained in N_{d-1-r} hyperplanes of S^*. Thus, S^* has order q. □

The existence of projective spaces follows from the existence of finite fields. In Theorem 5.14 and Theorem 5.15, we summarize known results for the existence of projective spaces.

Theorem 5.14 ([LN94])**.** *The order of every finite field is a prime power p^n and conversely, for every prime power p^n there exists a finite field of order p^n.*

Using Theorem 5.14, [Kåh02] shows that for every prime power $q = p^n$ there exists a d-dimensional projective space of order q. It is an open conjecture that the order of any projective space is a prime power.

Theorem 5.15 ([Kåh02])**.** *Let $q = p^n$ be a prime power and $d \geq 2$ be an integer. Then there exists a d-dimensional projective space of order q.*

The proof of Theorem 5.15 in [Kåh02] is constructive. If F is a field of order $q = p^n$ then let V be a $(d+1)$-dimensional vector space over F. The set of points S is defined to be the set of all 1-dimensional subspaces of V. The h-dimensional subspaces of S are defined to be the $(h+1)$-dimensional subspaces of F. This construction satisfies the axioms of Definition 5.8.

Definition 5.16. *Let $d \geq 2$ and $q > 1$ be integers and let S be a d-dimensional projective space of order q. A hypergraph $H = (V, E)$ is a* truncated d-dimensional projective space of order q *if there exists a point $P \subseteq S$ such that*

$$V = S \setminus P \quad \textit{and} \quad E = \{U \mid U \textit{ hyperplane in } S \textit{ and } P \cap U = \emptyset\} \ .$$

Lemma 5.17. *Let $H = (V, E)$ be a truncated d-dimensional projective space of order q. Then, H is an m-uniform m-partite q^{d-1}-regular hypergraph with q vertices per part, where $m = q^{d-1} + q^{d-2} + \cdots + q + 1$. Moreover, each d edges in E have at least one vertex in common.*

Proof. Let S be the corresponding projective space of dimension d and order q and P be the point in S that is chosen in Definition 5.16. By Lemma 5.12 and Lemma 5.13, the number of incident lines of P is $m = q^{d-1} + q^{d-2} + \cdots + q + 1$. Let $l_1, l_2, \ldots, l_m$ be the lines incident to P. The intersection of each two distinct incident lines of P is exactly the point P. By Lemma 5.11, for each hyperplane U in S not containing the point P and each line incident to P, there exists a unique point in the intersection of P and U. We define the parts of the hypergraph H by $V_i = l_i \setminus P$ for $i = 1, 2, \ldots, m$. By the arguments above, it is obvious that H is m-uniform m-partite with parts $V = V_1 \dot\cup \cdots \dot\cup V_m$. By Lemma 5.12, each line incident to P contains exactly $q + 1$ points. Thus, the parts of H have size q.

Let Q be a point in S distinct from P. We count the number of hyperplanes incident to Q but not P. Consider the dual projective space S^*. Then, P^* and Q^* are hyperplanes in S^* and we have to count the number of points in $Q^* \setminus P^*$. The subspace $Q^* \cap P^*$ has dimension $d - 2$ since

$$\dim(Q^* \cap P^*) = \dim(Q^*) + \dim(P^*) - \dim(\langle P^*, Q^* \rangle) = (d-1) + (d-1) - d = d - 2 \ .$$

Thus,

$$\begin{aligned} |Q^* \setminus P^*| = |Q^*| - |Q^* \cap P^*| &= \left(q^{d-1} + q^{d-2} + \cdots + q + 1\right) - \left(q^{d-2} + q^{d-3} + \cdots + q + 1\right) \\ &= q^{d-1} \ . \end{aligned}$$

Thus, in S there are q^{d-1} hyperedges incident to Q but not to P. Then, the hypergraph H is q^{d-1}-regular.

Let $U_1, U_2, \ldots, U_k$ be k hyperplanes in S. We prove that $\dim(U_1 \cap U_2 \cap \cdots \cap U_k) \geq d - k$ by induction. For $k = 1$ we have $\dim(U_1) = d - 1$ since U_1 is a hyperplane in S. Let $k > 1$. Then, by the induction hypothesis,

$$\begin{aligned} \dim(U_1 \cap \cdots \cap U_k) &= \dim(U_1 \cap \cdots \cap U_{k-1}) + \dim(U_k) - \dim(\langle U_1 \cap \cdots \cap U_{k-1}, U_k \rangle) \\ &\geq (d - k - 1) + (d - 1) - d = d - k \ . \end{aligned}$$

With $k = d$ we have $\dim(U_1 \cap U_2 \cap \cdots \cap U_d) \geq 0$ and there exists at least one point in $U_1 \cap U_2 \cap \cdots \cap U_d$. Then, in the hypergraph H each d edges intersect in at least one vertex. $\square$

Theorem 5.18. *Let q be a prime power and let t be a positive integer. Let $m = q^t + \cdots + q + 1$ and $r = q^t$. Then, there exists an m-uniform m-partite r-regular hypergraph with q vertices per part such that every maximum t-shallow edge set has size t. This shows*

$$\eta_t(m) \leq \eta_t(H) \leq (m^{1/t} - 1)^{-1}$$

for the specified values of m.

Proof. Let $H = (V, E)$ be a truncated $(t+1)$-dimensional projective space of order q. This exists since q is a prime power. By Lemma 5.17, each $(t+1)$ distinct edges in H have a vertex in common. Thus, the maximum t-shallow edge set in H has size t. With $m = q^t + \cdots + q + 1 \leq (q+1)^t$, we have $q \geq m^{1/t} - 1$ and obtain

$$\eta_t(m) \leq \eta_t(H) = \frac{t}{qt} = \frac{1}{q} \leq \frac{1}{m^{1/t} - 1} .$$

$\square$

Theorem 5.19. *Let t be a positive integer. Then there exists an m-uniform m-partite 2^{t-1}-regular hypergraph that has no $(t-1)$-shallow hitting edge set, where $m = 2^t - 1$ respectively $t = \log_2(m+1)$.*

Proof. Let $H = (V, E)$ be a truncated t-dimensional projective space of order 2. It exists since 2 is a prime power. By Lemma 5.17, H is an m-uniform m-partite 2^{t-1}-regular hypergraph with 2 vertices per part, where $m = 2^{t-1} + 2^{t-2} + \cdots + 2 + 1 = 2^t - 1$. Moreover, each t edges have a vertex in common. Then, each hitting edge set M has size at least $t+1$ and shallowness at least t. Thus, H is a hypergraph without a $(t-1)$-shallow hitting edge set where

$$t = \log_2(m+1) .$$

$\square$

The result in Theorem 5.19 improves the lower bound in Theorem 3.24 and shows

$$t(m) \geq \lfloor \log_2(m+1) \rfloor$$

for the least integer $t = t(m)$ such that every m-uniform regular hypergraph has a t-shallow hitting edge set. Additionally, this construction is m-partite.

6. NP-Completeness

For positive integers $m \geq 2$, t and r, we define the problem ShallowHittingEdgeSet$[m, t, r]$ as follows. Given an m-uniform m-partite r-regular hypergraph H, the problem is to decide whether H has a t-shallow hitting edge set. Note that m, t and r are part of the problem and not part of the input. Note that, by the definition of hypergraphs in Section 2.2, we allow multiple edges, i.e. the edge set E is a multiset. In this chapter, we show that for

$$t_{\max}(m) = \frac{1}{2}\log_2(m) - \mathcal{O}(\log\log(m)) \ ,$$

the problem ShallowHittingEdgeSet$[m, t, r]$ is $\mathcal{NP}$-complete for all integers $m \geq 3$ and all positive integers t with $t \leq t_{\max}(m)$ and $r = 2t$.

Karp showed in [Kar72] that the 3D-Matching problem is $\mathcal{NP}$-complete. The 3D-Matching problem is to decide, given a 3-uniform 3-partite hypergraph H, whether there exists a perfect matching in H. In [Boo80] it is claimed that ShallowHittingEdgeSet$[3, 1, 3]$ is $\mathcal{NP}$-complete, but we found no proof. Therefore, we prove the $\mathcal{NP}$-completeness of ShallowHittingEdgeSet$[3, 1, 3]$ in Lemma 6.1. In the next step, we provide a reduction to ShallowHittingEdgeSet$[3, 1, r]$, for all integers $r \geq 3$. Then, we show the $\mathcal{NP}$-completeness of ShallowHittingEdgeSet$[m, t, r]$ for all positive integers $m \geq 3$ and all integers t with $1 \leq t \leq t_{\max}(m)$ and $r = 2t$.

Lemma 6.1. *The problem* ShallowHittingEdgeSet$[3, 1, 3]$ *is $\mathcal{NP}$-complete.*

Proof. Clearly, the problem ShallowHittingEdgeSet$[3, 1, 3]$ is in $\mathcal{NP}$. For this, we can non-deterministically generate an edge set M and then check in polynomial time if M is a perfect matching.

Next, we show that ShallowHittingEdgeSet$[3, 1, 3]$ is $\mathcal{NP}$-hard by providing a reduction from 3D-Matching. Given a 3-uniform 3-partite hypergraph $H = (V_1 \dot\cup V_2 \dot\cup V_3, E)$, we can assume that $|V_1| = |V_2| = |V_3|$ and that each vertex has at least one incident edge. Otherwise, we reduce H to a trivial no-instance.

First, we eliminate the vertices of degree 1. Let e be an edge that contains a vertex of degree 1. We remove all vertices in e and all incident edges of these vertices. By repeating this step, we obtain a 3-uniform 3-partite hypergraph $H' = (V_1' \dot\cup V_2' \dot\cup V_3', E')$ with no vertices of degree 1. Observe that it still holds that $|V_1'| = |V_2'| = |V_3'|$. It holds that H has a perfect

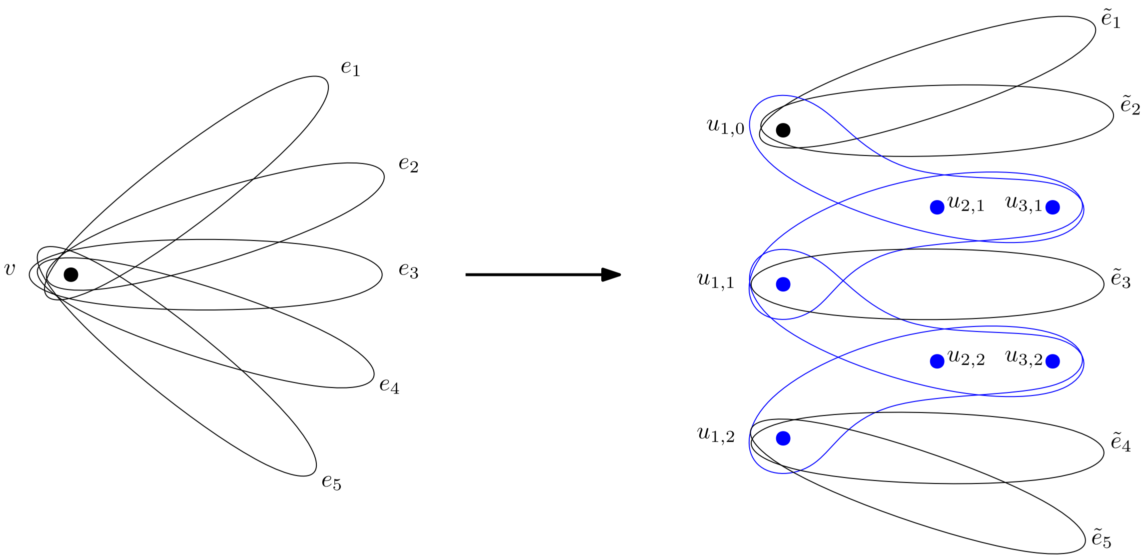

Figure 6.1: Reduction of a vertex v of degree $\deg(v) \geq 4$. This figure contains only the relevant vertices and edges. Additional vertices and the edges $\tilde{e}_j^{(0)}$ and $\tilde{e}_j^{(1)}$, $j = 1, 2, \ldots, d-3$, are colored blue.

matching if and only if H' has a perfect matching since every edge that contains a vertex of degree 1 must be in the perfect matching.

In the second step, we eliminate the vertices of degree greater than 3. Assume that v is a vertex of degree $\deg(v) \geq 4$ and without loss of generality $v \in V_1'$. Let $\operatorname{Inc}(v) = \{e_1, e_2, \ldots, e_d\}$ be the set of incident edges of v with $d = \deg(v)$. Define $u_{1,0} = v$. Then, we construct the hypergraph $\tilde{H} = (\tilde{V}, \tilde{E})$ with

$$\tilde{V} = V' \cup \{u_{i,j} \mid i = 1, 2, 3 \text{ and } j = 1, 2, \ldots, d-3\} \ .$$

We define $\tilde{e}_1 = e_1$, $\tilde{e}_j = (e_j \setminus \{v\}) \cup \{u_{1,j-2}\}$ for $j = 2, 3, \ldots, d-1$ and $\tilde{e}_d = (e_d \setminus \{v\}) \cup \{u_{1,d-3}\}$. Moreover, we define $\tilde{e}_j^{(0)} = \{u_{1,j-1}, u_{2,j}, u_{3,j}\}$ and $\tilde{e}_j^{(1)} = \{u_{1,j}, u_{2,j}, u_{3,j}\}$ for $j = 1, 2, \ldots, d-3$. Then, the edge set $\tilde{E}$ is defined by

$$\tilde{E} = (E' \setminus \operatorname{Inc}(v)) \ \cup \ \{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_d\} \ \cup \ \left\{\tilde{e}_j^{(0)}, \tilde{e}_j^{(1)} \mid j = 1, 2, \ldots, d-3\right\} \ .$$

The reduction is illustrated in Figure 6.1. Observe that the number of vertices v with $\deg(v) \geq 4$ in $\tilde{H}$ is less than in H'.

We have to show that H' has a perfect matching if and only if $\tilde{H}$ has a perfect matching. If H' contains a perfect matching M', then exactly one edge in $\{e_1, \ldots, e_d\}$ is in M'. Let e_k be this edge for some k. Then,

$$\begin{aligned}\tilde{M} =& (M' \setminus \{e_k\}) \ \cup \ \left\{\tilde{e}_j^{(0)} \mid j = 1, \ldots, \min\{k-2, d-3\}\right\} \ \cup \ \{\tilde{e}_k\} \ \cup \\ & \left\{\tilde{e}_j^{(1)} \mid j = \max\{1, k-1\}, \ldots, d-3\right\}\end{aligned}$$

is a perfect matching in $\tilde{H}$.

Now, let $\tilde{M}$ be a perfect matching in $\tilde{H}$. Then, since all vertices $u_{2,j}$, $j = 1, 2, \ldots, d-3$, must be covered exactly once, exactly $d-3$ edges of $\{\tilde{e}_j^{(0)}, \tilde{e}_j^{(1)} \mid j = 1, 2, \ldots, d-3\}$ are in $\tilde{M}$. Since all vertices $u_{1,j}$, $j = 0, 1, \ldots, d-3$ must be covered, there exists exactly one edge $\tilde{e}_k \in M$. Then, it must hold that

$$\left\{\tilde{e}_j^{(0)} \mid j = 1, \ldots, \min\{k-2, d-3\}\right\} \ \cup \ \left\{\tilde{e}_j^{(1)} \mid j = \max\{1, k-1\}, \ldots, d-3\right\} \subseteq \tilde{M} \ ,$$

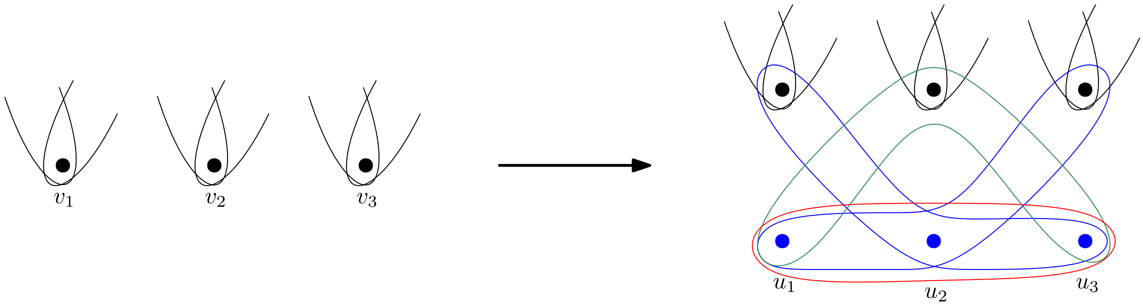

Figure 6.2: Reduction of three vertices $v_1 \in V_1''$, $v_2 \in V_2''$ and $v_3 \in V_3''$ of degree 2. This figure contains only the relevant vertices and edges. The additional vertices and edges are colored blue, green and red.

since otherwise $\tilde{M}$ would not be a perfect matching. Then,

$$M' = \left(\tilde{M} \setminus \left(\{\tilde{e}_j^{(0)}, \tilde{e}_j^{(1)} \mid j = 1, 2, \ldots, d-3\} \cup \{\tilde{e}_k\}\right)\right) \cup \{e_k\}$$

is a perfect matching in H'.

By repeating this reduction for all vertices of degree greater than 3, we obtain a 3-uniform 3-partite hypergraph $H'' = (V_1'' \dot\cup V_2'' \dot\cup V_3'', E'')$ with $2 \leq \deg(v) \leq 3$ for all vertices $v \in V''$, $V'' = V_1'' \dot\cup V_2'' \dot\cup V_3''$. Observe that each part of H'' contains the same number of vertices and thus, each part of V'' contains the same number of vertices of degree 2.

In the last step, we eliminate all vertices of degree 2. Let $v_1 \in V_1''$, $v_2 \in V_2''$ and $v_3 \in V_3''$ be three vertices of degree 2. We construct a hypergraph $\hat{H} = (\hat{V}, \hat{E})$ with

$$\hat{V} = V'' \cup \{u_1, u_2, u_3\} .$$

We define $\hat{e}_1 = \{v_1, u_2, u_3\}$, $\hat{e}_2 = \{u_1, v_2, u_3\}$, $\hat{e}_3 = \{u_1, u_2, v_3\}$ and $\hat{e}_4 = \{u_1, u_2, u_3\}$. We define $\hat{E}$ to be

$$\hat{E} = E'' \cup \{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\} .$$

The reduction is illustrated in Figure 6.2.

We have to show that H'' has a perfect matching if and only if $\hat{H}$ has a perfect matching. Let M'' be a perfect matching in H''. Then $\hat{M} = M'' \cup \{\hat{e}_4\}$ is a perfect matching in $\hat{H}$. On the other hand, let $\hat{M}$ be a perfect matching in $\hat{H}$. Since every two edges in $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ have a common vertex, at most one edge of $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ can be in $\hat{M}$. Since all three vertices u_1, u_2 and u_3 must be covered, the edge $\hat{e}_4$ must be in $\hat{M}$. Thus, no edge in $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is in $\hat{M}$. Then, $M'' = \hat{M} \setminus \{\hat{e}_4\}$ is a perfect matching in H''.

Since in every part of H' is the same number of vertices with degree 2, we can repeat this reduction until we obtain a 3-uniform 3-partite 3-regular hypergraph.

Since all reduction steps run in polynomial time and are repeated polynomial times, the overall reduction runs in polynomial time. Thus, we can summarize that the problem ShallowHittingEdgeSet$[3, 1, 3]$ is $\mathcal{NP}$-complete. $\square$

Lemma 6.2. *The problem* ShallowHittingEdgeSet$[3, 1, r]$ *is $\mathcal{NP}$-complete for all $r \geq 3$.*

Proof. We prove this lemma by induction over r. Clearly, ShallowHittingEdgeSet$[3, 1, r]$ is in $\mathcal{NP}$ for all r. For $r = 3$, Lemma 6.1 shows the $\mathcal{NP}$-completeness of the problem ShallowHittingEdgeSet$[3, 1, 3]$. Thereby, let $r > 3$ and assume that the problem

ShallowHittingEdgeSet$[3, 1, r-1]$ is $\mathcal{NP}$-complete. We show the $\mathcal{NP}$-completeness of ShallowHittingEdgeSet$[3, 1, r]$ by a reduction from ShallowHittingEdgeSet$[3, 1, r-1]$.

Let $H = (V, E)$ be a 3-uniform 3-partite $(r-1)$-regular hypergraph with $V = V_1 \dot\cup V_2 \dot\cup V_3$. It must hold that $|V_1| = |V_2| = |V_3|$ since otherwise we can reduce H to a trivial no-instance. Let $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$ three vertices with degree $r-1$. Then, we construct a hypergraph $\hat{H} = (\hat{V}, \hat{E})$ with the set of vertices $\hat{V} = V \cup \{u_1, u_2, u_3\}$. Let $\hat{e}_1 = \{v_1, u_2, u_3\}$, $\hat{e}_2 = \{u_1, v_2, u_3\}$, $\hat{e}_3 = \{u_1, u_2, v_3\}$ and $\hat{e}_4 = \{u_1, u_2, u_3\}$. We define the multiset of edges to be

$$\hat{E} = E \ \cup \ \{\hat{e}_1, \hat{e}_2, \hat{e}_3, \underbrace{\hat{e}_4, \ldots, \hat{e}_4}_{(r-2)\text{ times}}\} \ .$$

Clearly, $\hat{H}$ is a 3-uniform 3-partite hypergraph and has less vertices of degree $r-1$ than H.

If H has a perfect matching M, then $\hat{M} = M \cup \{\hat{e}_4\}$ is a perfect matching in $\hat{H}$. On the other hand, assume that $\hat{H}$ has a perfect matching $\hat{M}$. Then at most one edge of $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ can be in $\hat{M}$ since every two edges of $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ have a common vertex. Since all three vertices u_1, u_2 and u_3 must be covered by $\hat{M}$, one edge $\hat{e}_4$ must be in $\hat{M}$. But then, no edge of $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ can be in $\hat{M}$, since $e_i \cap e_j \neq \emptyset$ for $i, j = 1, 2, 3, 4$. Then, $M = \hat{M} \setminus \{\hat{e}_4\}$ is a perfect matching in H.

By repeating this reduction step, we obtain a 3-uniform 3-regular r-regular hypergraph H' with multiple edges that has a perfect matching if and only if H has a perfect matching. The reduction runs in polynomial time. This shows the $\mathcal{NP}$-completeness of ShallowHittingEdgeSet$[1, 3, r]$. □

We now prove that the problem ShallowHittingEdgeSet$[m, t, r]$ is $\mathcal{NP}$-complete for $m \geq 3$ and all positive integers $t \leq t_{\max}(m)$ and $r = 2t$. For this, we consider the dual problem. We say that a hypergraph is *proper edge-colorable with k colors* if there exists a function $\chi\colon E \to \{1, 2, \ldots, k\}$ such that for every vertex, no two incident edges have the same color. Recall that a t-shallow vertex cover in a hypergraph $H = (V, E)$ is a set $V' \subseteq V$ of vertices such that each edge contains between 1 and t vertices of V', i.e. $1 \leq |e \cap V'| \leq t$ for all $e \in E$. For positive integers r, t and $m \geq 2$, we define the problem ShallowVertexCover$[m, t, r]$ as follows. Given an m-regular r-uniform hypergraph with a proper edge-coloring with m colors, does there exist a t-shallow vertex cover? Note that, if ShallowVertexCover$[m, t, r]$ is $\mathcal{NP}$-complete, then ShallowHittingEdgeSet$[m, t, r]$ is $\mathcal{NP}$-complete. To see this, let H be an m-regular r-uniform hypergraph with a proper edge-coloring with m colors. Then, the dual hypergraph H^* is m-uniform m-partite r-regular and has a t-shallow hitting edge set if and only if H has a t-shallow vertex cover. Thus, it is sufficient to prove the $\mathcal{NP}$-completeness of ShallowVertexCover$[m, t, r]$ for the specified values of m, t and r. First, we give a construction of a hypergraph that is used as gadget in Theorem 6.5. The construction is motivated by the construction in Theorem 3.24.

Lemma 6.3. *Let $t \geq 2$ be an integer and $m = \binom{2t-1}{t-1} = \binom{2t}{t}/2$. There exists an m-regular $2t$-uniform hypergraph $\tilde{H} = (\tilde{V}, \tilde{E})$ with a proper edge-coloring with m colors such that*

1. *$\tilde{V} = \tilde{U}_1 \cup \tilde{U}_2 \cup \cdots \cup \tilde{U}_{2t}$ and $\tilde{U}_p = \{\tilde{u}_{p,0}, \tilde{u}_{p,1}\}$ for $p = 1, 2, \ldots, 2t$.*
2. *if M is a vertex cover with $\tilde{U}_1 \subseteq M$ then M is not t-shallow.*
3. *for every $\tilde{M} \subseteq \tilde{U}_1$ with $|\tilde{M}| \leq 1$, $M = \tilde{M} \cup \{\tilde{u}_{p,0} \mid p = 2, 3, \ldots, 2t\}$ is a t-shallow vertex cover.*

Proof. Let $\tilde{V}$ be the vertex set of $\tilde{H}$ defined in Property 1. We define the edge set $\tilde{E}$ to be

$$\tilde{E} = \left\{ \bigcup_{p \in I} \tilde{U}_p \ \middle| \ I \in \binom{[2t]}{t} \right\} \ .$$

Obviously, $\tilde{H}$ is $2t$-uniform. Each vertex has exactly $\binom{2t-1}{t-1}$ incident edges. Thus, $\tilde{H}$ is m-regular with $m = \binom{2t-1}{t-1}$. To see that $\tilde{H}$ is proper edge-colorable with m colors, observe that for every $e \in \tilde{E}$, $\bar{e} := \tilde{V} \setminus e$ is an edge in $\tilde{E}$. We define the edge-coloring $\chi\colon \tilde{E} \to \{1, 2, \ldots, m\}$ such that exactly the edges e and $\bar{e}$ receive the same color. For this coloring, we need $\binom{2t}{t}/2 = \binom{2t-1}{t-1}$ colors. Moreover, no vertex has two incident edges of the same color.

Let M be a vertex cover with $\tilde{U}_1 \subseteq M$. Let N be the number of sets $\tilde{U}_p$ with $|\tilde{U}_p \cap M| \geq 1$. If $N \leq t$ then M is not a vertex cover. If $N > t$ then M is not t-shallow. Thus, Property 2 is fulfilled. Let $\tilde{M} \subseteq \tilde{U}_1$ with $|\tilde{M}| \leq 1$. Then, $M = \tilde{M} \cup \{\tilde{u}_{p,0} \mid p = 2, 3, \ldots, 2t\}$ is a t-shallow vertex cover. Thus, Property 3 is fulfilled. □

We need the following definitions. A *spanning subgraph* of a graph G is a subgraph of G with the same vertex set as G. A *k-factor* of a graph G is a spanning k-regular subgraph of G. A *k-factorization* of a graph G is a partition of the edge set into disjoint k-factors of G. For the proof of Theorem 6.5, we need the following lemma.

Lemma 6.4 ([MR85]). *Let n be a positive integer. The complete graph K_{2n} has a 1-factorization.*

Proof. Let $V = \mathbb{Z}_{2n-1} \cup \{\infty\}$ be the vertex set of the complete graph. We first define the 1-factor $M_0 = \{\{j, -j\} \mid j \in \mathbb{Z}_n \setminus \{0\}\} \cup \{0, \infty\}$. Moreover, we define $M_i = i + M_0 = \{\{i+j, i-j\} \mid j \in \mathbb{Z}_n \setminus \{0\}\} \cup \{i, \infty\}$ for $i = 0, 1, \ldots, 2n-1$. Obviously, the subgraphs (V, M_i), $i = 0, 1, \ldots, n-1$, are 1-factors of the complete graph. Moreover, each edge $\{u, v\}$ with $u, v \in V$ and $u \neq v$ is contained in a set M_i. Thus, $\{M_0, M_1, \ldots, M_{2n-2}\}$ is a 1-factorization of the complete graph. □

Theorem 6.5. *The problem* ShallowVertexCover$[m, t, r]$ *is $\mathcal{NP}$-complete for all positive integers $m \geq 3$, $t \leq t_{max}(m)$ and $r = 2t$ with*

$$t_{max}(m) = \frac{1}{2}\log_2(m) - \mathcal{O}(\log\log(m)) \ .$$

Proof. Note that, by Lemma 6.2, the problem ShallowVertexCover$[3, 1, r]$ is $\mathcal{NP}$-complete for all integers $r \geq 3$. For the specified values of m and t, we give a reduction from ShallowVertexCover$[3, 1, 2t]$ to ShallowVertexCover$[m, t, 2t]$. Clearly, the decision problem ShallowHittingEdgeSet$[m, t, r]$ is in $\mathcal{NP}$ for all positive integers $m \geq 2$, t and r.

Let $H_0 = (V_0, E_0)$ be a 3-regular $2t$-uniform hypergraph with a proper edge-coloring with three colors such that $E_{0,1}, E_{0,2}, E_{0,3}$ are the color classes. Let $n_0 = |E_{0,1}| = |E_{0,2}| = |E_{0,3}|$ be the number of edges per color class. We reduce H_0 to the hypergraph $H = (V, E)$ with

$$V = V_0 \cup \tilde{V} \quad \text{with} \quad \tilde{V} = \bigcup_{i=1}^{n_0} \bigcup_{l=1}^{t} \tilde{V}_{i,l} \ , \quad \tilde{V}_{i,l} = \tilde{U}_2^{(i,l)} \cup \tilde{U}_3^{(i,l)} \cup \cdots \cup \tilde{U}_{2t}^{(i,l)} \ ,$$
$$\tilde{U}_p^{(i,l)} = \left\{\tilde{u}_{p,0}^{(i,l)}, u_{p,1}^{(i,l)}\right\}$$

for $p = 2, 3, \ldots, 2t$ and $i = 1, 2, \ldots, n_0$ and $l = 1, 2, \ldots, t$. Moreover, we define the edge set E to be

$$E = E_0 \cup E_1 \cup \tilde{E} \ ,$$

where E_0 is the edge set from the hypergraph H_0, and E_1 and $\tilde{E}$ are defined below. Let $\tilde{V} = P_1 \dot{\cup} \ldots \dot{\cup} P_{(2t-1)n_0}$ be a partition of $\tilde{V}$ into parts of size $|P_1| = \cdots = |P_{(2t-1)n_0}| = 2t$

such that $|P_q \cap \tilde{U}_p^{(i,l)}| \in \{0,2\}$ for all $q = 1,2,\ldots,(2t-1)n_0$ and $p = 2,3,\ldots,2t$ and $i = 1,2,\ldots,n_0$ and $l = 1,2,\ldots,t$. Let E_1 be the multiset

$$E_1 = \{3 \cdot P_1, 3 \cdot P_2, \ldots, 3 \cdot P_{(2t-1)n_0}\} .$$

The hypergraph $H_1 = (V, E_0 \cup E_1)$ is 3-regular $2t$-uniform and has a proper edge coloring with three colors.

We now define the set $\tilde{E}$ that ensures that H has a t-shallow vertex cover if and only if H_0 has a 1-shallow vertex cover. Let $E_{0,k} = \{e_{k,1}, e_{k,2}, \ldots, e_{k,n_0}\}$ be the set of edges of H_0 of color k, $k = 1,2,3$. Since H_0 is $2t$-uniform, for each edge $e \in E_0$ it holds that $|e| = 2t$. By Lemma 6.4, the complete graph on the $2t$ vertices of e has a 1-factorization. Let $\mathcal{F}_{k,i} = \{F_{k,i,j} \mid j = 1,2,\ldots,2t-1\}$ the corresponding set of edge sets of such a 1-factorization of the complete graph on the vertices of $e_{k,i}$. Let $F_{k,i,j} = \{f_{k,i,j,l} \mid l = 1,2,\ldots,t\}$ be the set of edges of a 1-factor of the 1-factorization, $j = 1,2,\ldots,2t-1$. That is, each $f_{k,i,j,l}$ is a pair of vertices in the hypergraph H_0. Moreover, for each pair of adjacent vertices there is a set $f_{k,i,j,l}$ that contains exactly these vertices. Additionally, for all $k = 1,2,3$ and $j = 1,2,\ldots,2t-1$ it holds that

$$V_0 = \bigcup_{i=1}^{n_0} \bigcup_{l=1}^{t} f_{k,i,j,l} . \tag{6.1}$$

We define the hypergraph $\tilde{H}_{k,i,j,l} = (\tilde{V}_{k,i,j,l}, \tilde{E}_{k,i,j,l})$ to be the hypergraph from Lemma 6.3 over the vertex set $\tilde{V}_{k,i,j,l} = f_{k,i,j,l} \cup \tilde{V}_{i,l}$. That is, $f_{k,i,j,l}$ corresponds to the set $\tilde{U}_1$ in Lemma 6.3 and each $\tilde{U}_p^{(i,l)} \subseteq \tilde{V}_{i,l}$ corresponds to the set $\tilde{U}_p$ in Lemma 6.3. Then, we define the edge set $\tilde{E}$ to be

$$\tilde{E} = \bigcup_{k=1}^{3} \bigcup_{j=1}^{2t-1} \tilde{E}_{k,j} \quad \text{with} \quad \tilde{E}_{k,j} = \bigcup_{i=1}^{n_0} \bigcup_{l=1}^{t} \tilde{E}_{k,i,j,l} .$$

Since each edge in $\tilde{E}_{k,i,j,l}$ has size $2t$ and H_1 is $2t$-uniform, H is $2t$-uniform. Let $\tilde{m} = \binom{2t}{t}/2$ and $m = 3 + 3(2t-1)\tilde{m}$. Let $\tilde{H}_{k,j} = (V, \tilde{E}_{k,j})$. We show that each $\tilde{H}_{k,j}$ is $\tilde{m}$-regular and has a proper edge coloring with $\tilde{m}$ colors. Then, it follows that $H = (V, E_0 \cup E_1 \cup \tilde{E})$ is m-regular and has a proper edge coloring with m colors. Observe that $\tilde{H}_{k,i,j,l}$ and $H_{k,i',j,l'}$ have different vertex sets if $i \neq i'$ or $l \neq l'$. By Equation 6.1, each vertex in V_0 is covered by an edge in $\tilde{E}_{k,j}$, for all $k = 1,2,3$ and $j = 1,2\ldots,2t-1$. It follows that each vertex in $V = V_0 \cup \tilde{V}$ is covered by an edge in $\tilde{E}_{k,j}$, for all $k = 1,2,3$ and $j = 1,2\ldots,2t-1$. By Lemma 6.3, each $\tilde{H}_{k,i,j,l}$ is $\tilde{m}$-regular and has a proper edge coloring with $\tilde{m}$ colors. It follows that each $\tilde{H}_{k,j}$ is $\tilde{m}$-regular and has a proper edge coloring with $\tilde{m}$ colors. Thus, H is m-regular and has a proper edge coloring with m colors.

We have to show that the hypergraph $H = (V, E)$ has a t-shallow vertex cover if and only if $H_0 = (V_0, E_0)$ has a 1-shallow vertex cover. Suppose that H_0 has a 1-shallow vertex cover M_0. Then, by Lemma 6.3,

$$M = M_0 \cup \{u_{p,0}^{(i,l)} \mid p = 2,3,\ldots,2t \text{ and } i = 1,2,\ldots,n_0 \text{ and } l = 1,2,\ldots,t\}$$

is a t-shallow vertex cover in H. If H_0 has no 1-shallow vertex cover, then every vertex cover M_0 in H_0 has an edge $e_{k,i}$ such that $|e_{k,i} \cap M_0| \geq 2$. Let $v_1, v_2 \in e_{k,i} \cap M_0$ be two distinct adjacent vertices in the vertex cover. Then, in the 1-factorization $\mathcal{F}_{k,i}$ there exists a 1-factor $F_{k,i,j}$ for some j such that $\{v_1, v_2\} = f_{k,i,j,l} \in F_{k,i,j}$ for some l. It holds that $f_{k,i,j,l} \subseteq M_0$. Consider the hypergraph $\tilde{H}_{k,i,j,l}$, that is a subgraph of H. By Lemma 6.3 Property 2, $\tilde{H}_{k,i,j,l}$ has no t-shallow vertex cover. Then, H has no t-shallow vertex cover.

It holds that

$$m = 3 + \frac{3}{2}(2t-1)\binom{2t}{t} \leq 3t4^t \ ,$$

since $\binom{2t}{t} \leq 4^t$. Thus, ShallowVertexCover$[t, m, 2t]$ is $\mathcal{NP}$-complete for all integers $t \geq 1$ and all integers $m \geq m_{\min}(t) = 3t4^t$. Let $W(x)$ be the Lambert W-function, i.e. the unique function defined by $x = W(x)\exp(W(x))$ for all real numbers $x \geq -e^{-1}$. It holds that [HH08] $W(x) = \ln x - \ln\ln x + o(1)$, where ln denotes the natural logarithm to the base e. Then,

$$t_{\max}(m) = \frac{1}{\ln(4)} W\left(\frac{m\ln(4)}{3}\right) = \frac{1}{2}\log_2(m) - \mathcal{O}(\log\log(m))$$

is the inverse function to $m_{\min}(t)$. □

7. Polychromatic Coloring of the Union of Strips

In defining geometric hypergraphs we follow [CU21] and [ACC$^+$11]. Let k be a positive integer. A coloring of the vertices of a hypergraph $H = (V, E)$ with k colors is a function $\chi\colon V \to \{1, 2, \ldots, k\}$. There are different kinds of vertex colorings of hypergraphs. A coloring of the vertices is called *proper* if each edge $e \in E$ contains at least two colors, i.e. $|\chi(e)| \geq 2$. In this chapter, we consider polychromatic colorings of the vertices. A coloring of the vertices with k colors is called *polychromatic* if each edge $e \in E$ contains all k colors, i.e. $|\chi(e)| = k$.

In this chapter, we are interested in finite sets of points in $\mathbb{R}^d$ for a dimension d. A *geometric range* R is a subset of $\mathbb{R}^d$. A *range capturing hypergraph* is a geometric hypergraph $\mathcal{H}(V, \mathcal{R})$ with a finite set of points $V \subseteq \mathbb{R}^d$ and a family $\mathcal{R}$ of ranges. The hypergraph $\mathcal{H}(V, \mathcal{R}) = (V, E)$ has vertex set V. A subset $e \subseteq V$ is an edge in E if and only if there exists a range $R \in \mathcal{R}$ such that $e = V \cap R$. In this case, we say that e is *captured* by the range R. We are particularly interested in the m-uniform subhypergraph $\mathcal{H}(V, \mathcal{R}, m)$ that consists of all edges in $\mathcal{H}(V, \mathcal{R})$ of size exactly m.

Given a range family $\mathcal{R}$ and a positive integer k, the integer $m = m_{\mathcal{R}}(k)$ is the smallest integer such that for every finite set of points $V \subseteq \mathbb{R}^d$ there exists a polychromatic vertex-coloring with k colors of $\mathcal{H}(V, \mathcal{R}, m)$. Clearly, $m_{\mathcal{R}}(k) \geq k$ since every hyperedge must contain k different colors. If for every positive integer m there exist a finite set of points $V \subseteq \mathbb{R}^d$ such that there exists no polychromatic k-coloring of $\mathcal{H}(V, \mathcal{R}, m)$, we define $m_{\mathcal{R}}(k) = \infty$.

We say that a range family $\mathcal{R}$ is *shrinkable* if for every finite set of points V, every positive integer m and every edge e in $\mathcal{H}(V, \mathcal{R}, m)$ there exists an edge e' in $\mathcal{H}(V, \mathcal{R}, m-1)$ with $e' \subseteq e$. In this section, we consider axis-aligned strips in d dimensions. This range family is defined by

$$\mathcal{R}_d = \bigcup_{i=1}^{d} \mathcal{R}_d^{(i)} \quad \text{where} \quad \mathcal{R}_d^{(i)} = \Big\{ \Big\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \ \Big|\ a \leq x_i \leq b \Big\} \ \Big|\ a, b \in \mathbb{R} \Big\} \ .$$

Observe that $\mathcal{R}_d^{(i)}$ is shrinkable for all positive integers d and $1 \leq i \leq d$. Moreover, observe that the union of shrinkable range families is shrinkable. Thus, $\mathcal{R}_d$ is shrinkable. When considering axis-aligned strips $\mathcal{R}_d$ in d dimensions, we assume that the points of the geometric hypergraph lie in general position. In this case, this means that $x_i \neq y_i$ for all points $(x_1, x_2, \ldots, x_d) \neq (y_1, y_2, \ldots, y_d)$ in V and all $i = 1, 2, \ldots, d$.

We define $m_d(k) = m_{\mathcal{R}}(k)$ for axis-aligned strips $\mathcal{R} = \mathcal{R}_d$ in d dimensions. As [ACC+11] pointed out, the problem of determining $m_d(k)$ for $\mathcal{R}_d$ can be seen purely combinatorial. That is, the problem of determining $m_d(k)$ is equivalent to the following problem. Given a finite set V of size n and d bijections $\pi_1, \ldots, \pi_d \colon V \to \{1, \ldots, n\}$, we have to color the set V in k colors such that for each bijection π_i, each $m_d(k)$ consecutive elements contain an element from each color. More formally, let k and d be positive integers. Then, $m_d(k)$ is the least integer such that for any finite set V of size n and any d bijections $\pi_1, \ldots, \pi_d \colon V \to \{1, 2, \ldots, n\}$, there exists a coloring of V with k colors such that each $m_d(k)$ consecutive elements of each bijection contain at least one element from each color.

First, we list some known results for $m_d(k)$ for axis-aligned strips in d dimensions.

- For $d = 1$ it is obvious that $m_d(k) = k$.
- For $d = 2$, it holds that $m_d(k) \leq 2k - 1$. This was proven in [ACC+11].
- If d is a positive integer, then $m_d(k) \leq k(4 \ln k + \ln d)$ is an upper bound proven in [ACC+11]. Thus, if d is a constant, then $m_d(k) \leq \mathcal{O}(k \log k)$.
- In [ACC+11] it is proven that
$$m_d(k) \geq 2 \cdot \left\lceil \frac{2d-1}{2d} \cdot k \right\rceil - 1$$
is a lower bound.
- By [PTT09], $m_d(k) \to \infty$ for $d \to \infty$.

In this section, we prove that $m_d(k) \leq \mathcal{O}(k)$ for axis-aligned strips in d dimensions, if d is a constant, with the method of shallow vertex covers. This is the best possible result (up to constant factors) since $m_d(k) \geq k$ and thus $m_d(k) \geq \Omega(k)$.

Lemma 7.1 ([ACC+11], [CU21]). *Let $\mathcal{R}$ be a shrinkable range family and suppose that every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ has a t-shallow vertex cover. Then, $m_{\mathcal{R}}(k) \leq (k-1)t + 1$.*

Proof. We prove this theorem by induction on k. For $k = 1$, all points in V receive the same color. This is a polychromatic 1-coloring of $\mathcal{H}(V, \mathcal{R}, m)$ for $m = 1$. Let $k \geq 2$ and let $m = (k-1)t + 1$. Let X be a t-shallow vertex cover of $H = \mathcal{H}(V, \mathcal{R}, m)$. Consider the hypergraph $H' = \mathcal{H}(V \setminus X, \mathcal{R}, m - t)$ with $m - t = (k-2)t + 1$. By the induction hypothesis there exists a polychromatic $(k-1)$-coloring of H', that is each hyperedge e' of H' receives all $k - 1$ colors. Since $\mathcal{R}$ is shrinkable, for every hyperedge e of H there exists a hyperedge e' of H' with $e' \subseteq e$. Define a coloring of the vertices of H by taking the coloring of H' and assign the k-th color to the vertices in X. Observe that this gives a polychromatic k-coloring of H. $\square$

Lemma 7.2. *Let $H = (V, E)$ be a hypergraph. Assume that there exist a multiset W of subsets of V and positive integers d, δ, Δ and t' such that*

1. *$|\{w \in W \mid v \in w\}| \leq d$ for all $v \in V$ and*
2. *$\delta \leq |w| \leq \Delta$ for all $w \in W$ and*
3. *for each edge $e \in E$ there exists a set $w \in W$ with $w \subseteq e$ and*
4. *for each edge $e \in E$ there exists a subset W' of W of size $|W'| \leq t'$ and $e \subseteq \bigcup_{w' \in W'} w'$.*

Then there exists a t-shallow vertex cover with $t = \mathrm{e}\mu d t' \cdot (1 + o_{\mu d}(1))$, where $\mu = \Delta/\delta$.

Proof. Let $\tilde{H} = (W, \tilde{E})$ be the hypergraph with vertex set W and the multiset of edges

$$\tilde{E} = \{\{w \in W \mid v \in w\} \mid v \in V\}$$

That is, each edge in $\tilde{E}$ corresponding to a vertex $v \in V$ is the set of all sets $w \in W$ containing v. By Property 1, $|\tilde{e}| \leq d$ for all edges $\tilde{e} \in \tilde{E}$. By Property 2, the minimum degree of $\tilde{H}$ is δ and the maximum degree of $\tilde{H}$ is Δ. Thus, $\tilde{H}$ is μ-near regular with $\mu = \Delta/\delta$. By Theorem 3.28, there exists a $\tilde{t}$-shallow hitting edge set $\tilde{M}$ in $\tilde{H}$, where $\tilde{t} = \mathrm{e}\mu d \cdot (1 + o_{\mu d}(1))$. Let X be the set of vertices in H that correspond to the edges in $\tilde{M}$. By Property 3, X is a hitting edge set in $\tilde{H}$ and thus a vertex cover in H. By Property 4, X is t-shallow with $t = t' \cdot \tilde{t}$. □

Theorem 7.3. *Let $\mathcal{R} = \mathcal{R}_d$ be the family of axis-aligned strips in d dimensions. Then,*

$$m_d(k) = m_{\mathcal{R}}(k) \leq 3\mathrm{e}kd(1 + o_d(1)) \ .$$

Thus, $m_{\mathcal{R}}(k) = \mathcal{O}(k)$ for constant d.

Proof. For a positive integer m, let $\mathcal{H}(V, \mathcal{R}, m) = (V, E)$ be a range-capturing hypergraph with point set V and range family $\mathcal{R}$ as defined above. First, assume that $|V| = rn$ for a positive integer n and $r = (m+1)/2$. We show that there exists a multiset W of subsets of V satisfying the properties of Lemma 7.2 with $\delta = \Delta = r$ and $t' = 3$ and d equal to the dimension. With that, we can conclude that there exists a t-shallow vertex cover with $t = 3\mathrm{e}d(1 + o_d(1))$. Thus,

$$m_d(k) \leq (k-1)t + 1 \leq (k-1) \cdot 3\mathrm{e}d(1 + o_d(1)) + 1 \leq 3\mathrm{e}kd(1 + o_d(1)) \ .$$

We construct the multiset W as follows. For $i = 1, 2, \ldots, d$, let $\pi_i \colon V \to \{1, \ldots, rn\}$ be the ordering of the points in V respective the strips of the i-th dimension. That is, $(\pi_i^{-1}(1))_i < (\pi_i^{-1}(2))_i < \cdots < (\pi_i^{-1}(rn))_i$. Then, for each edge e in $\mathcal{H}(V, \mathcal{R}_d^{(i)}, m)$, $\pi_i(e)$ contains m consecutive integers. For $i = 1, 2, \ldots, m$, define W_i to be

$$\begin{aligned} W_i = &\left\{\pi_i^{-1}\left(\{rj+1, rj+2, \ldots, (r+1)j\}\right) \mid j = 0, 1, \ldots, n-1\right\} \\ =&\{\{\pi_i^{-1}(1), \ldots, \pi_i^{-1}(r)\}, \{\pi_i^{-1}(r+1), \ldots, \pi_i^{-1}(2r)\}, \ldots, \\ &\{\pi_i^{-1}(r(n-1)+1), \ldots, \pi_i^{-1}(rn)\}\} \ . \end{aligned}$$

Observe that each W_i partitions the point set V. Define W to be the multiset union of all W_i, $i = 1, 2, \ldots, d$. That is, the multiplicity of an element w is the number of sets W_i with $w \in W_i$. Observe that W satisfies Property 1 of Lemma 7.2, where d the dimension. Moreover, note that $|w| = r$ for each $w \in W$. Since $r = (m+1)/2$, every edge of $\mathcal{H}(V, \mathcal{R}_d^{(i)}, m)$ completely contains an element of W_i. Thus, W satisfies Property 3 of Lemma 7.2. Moreover, for each edge e of $\mathcal{H}(V, \mathcal{R}_d^{(i)}, m)$ there exist three elements of W_i that cover the points of e. Thus, W satisfies Property 4 of Lemma 7.2 with $t' = 3$.

If $|V| = s \pmod r$ with $s \neq 0$, we add $r - s$ points such that for each $i = 1, 2, \ldots, d$, these points are the last $r - s$ points in the ordering π_i. Let Q be this set of $r - s$ additional points. That is, the points in Q are right and above the set V respective each coordinate i. We repeat the argument above for the geometric hypergraph $\tilde{H} = \mathcal{H}(V \cup Q, \mathcal{R}, m)$ and obtain a polychromatic k-coloring of $\tilde{H}$. Observe that $H = \mathcal{H}(V, \mathcal{R}, m)$ is a subgraph of $\tilde{H}$ and thus, every polychromatic k-coloring of $\tilde{H}$ gives a polychromatic k-coloring of H. □

The upper bound obtained in Theorem 7.3 proves an upper bound for the following problem, see Corollary 7.4. We consider point sets in two dimensions, i.e. $V \subseteq \mathbb{R}^2$ is a finite set of points and each range $R \in \mathcal{R}$ is a subset of $\mathbb{R}^2$. Let $\mathcal{R}_\alpha$ be the family of strips in two dimensions of direction $\alpha \in [0, \pi)$. More formally, we define $\mathcal{R}_\alpha$ to be

$$R_\alpha = \left\{ \left\{ (x, y) \in \mathbb{R}^2 \mid c \leq ax + by \leq d \right\} \mid a, b, c, d \in \mathbb{R},\ a \neq 0 \text{ or } b \neq 0,\ \tan \alpha = -a/b \right\} .$$

Here, we define $a/0 = \infty$ for $a > 0$, $a/0 = -\infty$ for $a < 0$ and $\tan(\pi/2) = \infty$. Let $\mathcal{R}$ be the union of strips of d directions $\mathcal{R} = \mathcal{R}_{\alpha(1)} \cup \cdots \cup \mathcal{R}_{\alpha(d)}$. We assume that the points in V lie in general position with respect to the directions $\alpha \in \{\alpha(1), \ldots, \alpha(d)\}$. In this case, this means that no two distinct points $(x_1, y_1) \neq (x_2, y_2) \in V$ lie on the same boundary of the same strip, i.e. $\tan \alpha \neq (y_2 - y_1)/(x_2 - x_1)$ for all directions $\alpha \in \{\alpha(1), \ldots, \alpha(d)\}$.

Clearly, each family $\mathcal{R}_{\alpha(i)} \subseteq \mathcal{R}$ for $i = 1, 2, \ldots, d$ induces an ordering of the points in V. Let $\pi_i \colon V \to \{1, 2, \ldots, n\}$ be this ordering for $i = 1, 2, \ldots, d$, where $n = |V|$ is the number of points and each π_i is a bijection. Then, we can reduce the set V of points (in two dimensions) to the following d-dimensional point set $V' \subseteq \mathbb{R}^d$

$$V' = \{(\pi_1(v), \pi_2(v), \ldots, \pi_d(v)) \mid v \in V\} .$$

Here, we consider the range family $\mathcal{R}_d$ of axis-aligned strips in d dimensions. Note that the underlying geometric hypergraphs are isomorphic, that is $\mathcal{H}(V, \mathcal{R}, m) \cong \mathcal{H}(V', \mathcal{R}_d, m)$ for all positive integers m. Thus, if $\mathcal{H}(V', \mathcal{R}_d, m)$ has a polychromatic k-coloring, then also $\mathcal{H}(V, \mathcal{R}, m)$. Thus, $m_{\mathcal{R}}(k) \leq m_d(k)$ for the union of strips of d directions $\mathcal{R} = \mathcal{R}_{\alpha(1)} \cup \cdots \cup \mathcal{R}_{\alpha(d)}$. The following corollary describes this result.

Corollary 7.4. *Let $\mathcal{R} = \mathcal{R}_{\alpha(1)} \cup \mathcal{R}_{\alpha(2)} \cup \cdots \cup \mathcal{R}_{\alpha(d)}$ be the union of strips of d directions in two dimensions. Then,*

$$m_{\mathcal{R}}(k) \leq m_d(k) \leq 3\mathrm{e}kd(1 + o_d(1)) .$$

Thus, $m_{\mathcal{R}}(k) = \mathcal{O}(k)$ for constant d.

8. Conclusion

In this thesis, we studied t-shallow (hitting) edge sets, which are a generalization of (perfect) matchings. In Section 3.1, we proved special cases of theorems from [BLV78] with the method of alternating paths. First, we derived a Hall-equivalence for the existence of t-shallow hitting edge sets in bipartite graphs in Theorem 3.3 and Corollary 3.4. Then, we generalized this theorem to graphs in Theorem 3.9 and Corollary 3.11. In particular, we proved for $t \geq 2$ that a graph G has a t-shallow hitting edge set if and only if $|X| \leq t|N(X)|$ for all independent sets X in G. In Theorem 3.6, we generalized Theorem 3.3 and we provided an equivalence for the existence of a weight function $c\colon E \to \mathbb{N} \cup \{0\}$ such that for each vertex v, the sum of all incident edge weights of v are at least $g(v)$ and at most $f(v)$. Here, $f, g\colon V \to \mathbb{N}$ are arbitrary functions with $1 \leq g(v) \leq f(v)$.

In Section 3.2, we provided sufficient conditions on the minimum degree $\delta'_{m-1}(H)$ (as defined in Section 3.2) for the existence of t-shallow hitting edge sets in an m-uniform m-partite hypergraph H with n vertices per part. In Theorem 3.20, we proved that if

$$\delta'_{m-1}(H) \geq \left\lceil \frac{n}{(m-1)t+1} \right\rceil + 1 \ ,$$

then there exists a t-shallow hitting edge set in H. On the other hand, there exists an m-uniform m-partite hypergraph H' with n vertices per part and

$$\delta'_{m-1}(H') \geq \left\lceil \frac{n}{(m-1)t+1} \right\rceil - 1$$

that has no t-shallow hitting edge set (Theorem 3.15). Thus, the condition described above is almost tight.

Question 8.1. Is $\delta'_{m-1}(H) \geq \lceil n/((m-1)t+1)\rceil$ sufficient for the existence of a t-shallow hitting edge set in an m-uniform m-partite hypergraph H? Or does there exist an m-uniform m-partite hypergraph H' with $\delta'_{m-1}(H') = \lceil n/((m-1)t+1)\rceil$ and no t-shallow hitting edge set?

Then, we used Theorem 3.20 to prove sufficient conditions on the minimum degree of an m-uniform hypergraph for the existence of t-shallow hitting edge sets. For that, we introduced the minimum degree $\delta_{m-1}(H)$ for an m-uniform hypergraph $H = (V, E)$. We proved that if $|V| = nm$ for an integer $n \geq n_0(m)$ and

$$\delta_{m-1}(H) \geq \frac{|V|}{(m-1)t+1} + \mathcal{O}(m^2\sqrt{n\log n})$$

then there exists a t-shallow hitting edge set. Contrary, there exists an m-uniform hypergraph $H' = (V', E')$ with

$$\delta_{m-1}(H') \geq \frac{|V'|}{(m-1)t+1} - 1$$

that has no t-shallow hitting edge set.

Question 8.2. Is $\delta_{m-1}(H) \geq n/((m-1)t+1)$ sufficient for the existence of a t-shallow hitting edge set in an m-uniform hypergraph H?

In Section 3.3, 3.4 and 3.5, we considered t-shallow hitting edge sets in m-uniform regular hypergraphs. Let $t = t(m)$ be the least integer such that every m-uniform regular hypergraph has a t-shallow hitting edge set. In Theorem 3.24, we proved $t(m) \geq \lfloor (1 + \log_2 m)/2 \rfloor$ by an explicit construction of an m-uniform m-partite regular hypergraph. This bound was improved in Theorem 5.19, where we proved $t(m) \geq \lfloor \log_2(m+1) \rfloor$. In Theorem 3.4, we proved $t(m) \leq \mathrm{e}m(1 + o(1))$ using the Lovász Local Lemma. Since the Lovász Local Lemma is constructive, there exists a randomized algorithm that finds a t-shallow hitting edge set with $t \leq \mathrm{e}m(1+o(1))$ in expected polynomial time. We described that algorithm in Section 3.7.

Question 8.3. Can you determine the exact value of $t(m)$?

Question 8.4. Does there exist a deterministic algorithm that outputs a t-shallow hitting edge set for $t = t(m)$?

In Chapter 4, we considered maximum t-shallow edge sets in m-uniform m-partite regular hypergraphs. We showed that every m-uniform m-partite regular hypergraph with parts of size n has a t-shallow hitting edge set of size

$$\frac{nt}{\mathrm{e}m^{1/t}}(1 - o_t(1)) \ ,$$

when $t \leq \Delta(H) = \delta(H)$. In Section 4.3, we showed that this result is tight up to the constant factor e^{-1}. Moreover, we showed that there exist $\mathcal{O}(m^2)$ matchings such that their union is a hitting edge set.

In Chapter 5, we provided an explicit construction of an m-uniform m-partite regular hypergraph through projective spaces. The construction was motivated by the extremal hypergraphs for Ryser's Conjecture, i.e. truncated projective planes. Using this construction, we proved that for every t there exist infinitely many m-uniform m-partite regular hypergraphs with parts of size n such that every t-shallow edge set has size at most

$$\frac{nt}{m^{1/t} - 1} \ .$$

Moreover, using this construction we showed $t(m) \geq \lfloor \log_2(m+1) \rfloor$ for $t(m)$ as defined above.

In Chapter 6, we discussed the $\mathcal{NP}$-completeness of deciding whether a given m-uniform m-partite r-regular hypergraph has a t-shallow hitting edge set. For this, we defined ShallowHittingEdgeSet$[m, t, r]$ to be the problem to decide whether such a hypergraph has a t-shallow hitting edge set. Clearly, for $t \geq t(m)$ this problem is decidable in $\mathcal{O}(1)$ since every m-uniform m-partite regular hypergraph has a $t(m)$-shallow hitting edge set. In this chapter, we showed that ShallowHittingEdgeSet$[m, t, r]$ is $\mathcal{NP}$-complete for all $m \geq 3$ and all positive integers $t \leq t_{\max}(m)$ and $r = 2t$, where

$$t_{\max}(m) = \frac{1}{2}\log_2(m) - \mathcal{O}(\log \log m) \ .$$

The reduction was from the $\mathcal{NP}$-complete problem 3D-Matching and used the construction from Theorem 3.24 as gadget. It is open whether the construction from Theorem 5.19 can be used to improve $t_{\max}(m)$ by the factor 2.

Question 8.5. Can you improve $t_{\max}(m)$ for $m \geq 3$, i.e. the largest integer $t = t_{\max}(m)$ such that ShallowHittingEdgeSet$[m, t', r]$ is $\mathcal{NP}$-complete for all integers t' with $1 \leq t' \leq t$ and some positive integer r?

In Chapter 7, we used the existence of shallow hitting edge sets in uniform regular hypergraphs to show Theorem 7.3. That is, $m_d(k)$ is the least integer such that given a finite set of points V in $\mathbb{R}^d$, we can color these points with k colors such that each axis-aligned strip that contains at least $m_d(k)$ points has a point from each color. Here, we proved $m_d(k) \leq 3\mathrm{e}kd(1 + o_d(1))$ and herewith improved a bound from [ACC+11] for constant d.

In the last paragraph, we state a problem we were not able to solve in this thesis. In the general case, there exists no equivalence for the existence of perfect matchings in hypergraphs. However, [CCKVk96] and [HT02] proved a Hall-type equivalence for the existence of perfect matchings in balanced hypergraphs. We use the notion of (totally) balanced hypergraphs from [Ber89] and [HT02]. Let $H = (V, E)$ be a hypergraph. A sequence $P = v_0 e_1 v_1 e_2 v_2 \ldots e_l v_l$ is called a *cycle* if $l \geq 3$ and $v_0 = v_l$ and $v_0, v_1, \ldots, v_{l-1}$ are pairwise distinct and $v_{i-1}, v_i \in e_i$ for $i = 1, 2, \ldots, l$. We define $V(P) = \{v_0, v_1, \ldots, v_l\}$ to be the vertices of the cycle and $E(P) = \{e_1, e_2, \ldots, e_l\}$ to be the edges of the cycle. A hypergraph is called *balanced* if every odd cycle has an edge containing three vertices of the cycle. A hypergraph is called *totally balanced* if every cycle of length at least three has an edge containing three vertices of the cycle. Note that balanced 2-uniform hypergraphs are bipartite and totally balanced 2-uniform hypergraphs are forests. There exists an generalization of Hall's Theorem to balanced hypergraphs. While [CCKVk96] gave a proof using the theory of linear programming, [HT02] gave a combinatorial proof.

Theorem 8.6 ([CCKVk96], [HT02])**.** *A balanced hypergraph $H = (V, E)$ has a perfect matching if and only if for all disjoint $A, B \subseteq V$ with $|A| > |B|$ there exists an edge $e \in E$ with $|e \cap A| > |e \cap B|$.*

Question 8.7. Can you generalize Theorem 8.6 to t-shallow hitting edge sets in (totally) balanced hypergraphs?

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